ACTUARIAL *Mathematics*

NEWTON
GERBER
HICKMAN
JONES
NESBITT

Second Edition

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AUTHORS' BIOGRAPHIES

NEWTON L. BOWERS, JR., Ph.D., F.S.A., M.A.A.A., received a B.S. from Yale University and a Ph.D. from the University of Minnesota. After his graduate studies, he joined the faculty at the University of Michigan and later moved to Drake University. He held the position of Ellis and Nelle Levitt Professor of Actuarial Science until his retirement in 1996.

HANS U. GERBER, Ph.D., A.S.A., received the Ph.D. degree from the Swiss Federal Institute of Technology. He has been a member of the faculty at the Universities of Michigan and Rochester. Since 1981 he has been on the faculty at the Business School, University of Lausanne, where he is currently head of the Institute of Actuarial Science.*

JAMES C. HICKMAN, Ph.D., F.S.A., A.C.A.S., M.A.A.A., received a B.A. from Simpson College and a Ph.D. from the University of Iowa. He was a member of the faculty at the University of Iowa and at the University of Wisconsin, Madison, until his retirement in 1993. For five years he was Dean of the School of Business at Wisconsin.

DONALD A. JONES, Ph.D., F.S.A., E.A., M.A.A.A., received a B.S. from Iowa State University and M.S. and Ph.D. degrees from the University of Iowa. He was a member of the actuarial faculty at the University of Michigan and a working partner of Ann Arbor Actuaries until 1990 when he became Director of the Actuarial Science Program at Oregon State University.

CECIL J. NESBITT, Ph.D., F.S.A., M.A.A.A., received his mathematical education at the University of Toronto and the Institute for Advanced Study in Princeton. He taught actuarial mathematics at the University of Michigan from 1938 to 1980. He served the Society of Actuaries from 1985 to 1987 as Vice-President for Research and Studies.*

*Professors Gerber and Nesbitt were involved as consultants with the revisions incorporated in the second edition.

AUTHORS' INTRODUCTIONS AND GUIDE TO STUDY

Introduction to First Edition*

This text represents a first step in communicating the revolution in the actuarial profession that is taking place in this age of high-speed computers. During the short period of time since the invention of the microchip, actuaries have been freed from numerous constraints of primitive computing devices in designing and managing insurance systems. They are now able to focus more of their attention on creative solutions to society's demands for financial security.

To provide an educational basis for this focus, the major objectives of this work are to integrate life contingencies into a full risk theory framework and to demonstrate the wide variety of constructs that are then possible to build from basic models at the foundation of actuarial science. Actuarial science is ever evolving, and the procedures for model building in risk theory are at its forefront. Therefore, we examine the nature of models before proceeding with a more detailed discussion of the text.

Intellectual and physical models are constructed either to organize observations into a comprehensive and coherent theory or to enable us to simulate, in a laboratory or a computer system, the operation of the corresponding full-scale entity. Models are absolutely essential in science, engineering, and the management of large organizations. One must, however, always keep in mind the sharp distinction between a model and the reality it represents. A satisfactory model captures enough of reality to give insights into the successful operation of the system it represents.

The insurance models developed in this text have proved useful and have deepened our insights about insurance systems. Nevertheless, we need to always keep

*Chapter references and nomenclature have been changed to be in accord with the second edition. These changes are indicated by italics.

before us the idea that real insurance systems operate in an environment that is more complex and dynamic than the models studied here. Because models are only approximations of reality, the work of model building is never done; approximations can be improved and reality may shift. It is a continuing endeavor of any scientific discipline to revise and update its basic models. Actuarial science is no exception.

Actuarial science developed at a time when mathematical tools (probability and calculus, in particular), the necessary data (especially mortality data in the form of life tables), and the socially perceived need (to protect families and businesses from the financial consequences of untimely death) coexisted. The models constructed at the genesis of actuarial science are still useful. However, the general environment in which actuarial science exists continues to change, and it is necessary to periodically restate the fundamentals of actuarial science in response to these changes.

We illustrate this with three examples:

1. The insurance needs of modern societies are evolving, and, in response, new systems of employee benefits and social insurance have developed. New models for these systems have been needed and constructed.
2. Mathematics has also evolved, and some concepts that were not available for use in building the original foundations of actuarial science are now part of a general mathematics education. If actuarial science is to remain in the mainstream of the applied sciences, it is necessary to recast basic models in the language of contemporary mathematics.
3. Finally, as previously stated, the development of high-speed computing equipment has greatly increased the ability to manipulate complex models. This has far-reaching consequences for the degree of completeness that can be incorporated into actuarial models.

This work features models that are fundamental to the current practice of actuarial science. They are explored with tools acquired in the study of mathematics, in particular, undergraduate level calculus and probability. The proposition guiding Chapters *1–14* is that there is a set of basic models at the heart of actuarial science that should be studied by all students aspiring to practice within any of the various actuarial specialities. These models are constructed using only a limited number of ideas. We will find many relationships among those models that lead to a unity in the foundations of actuarial science. These basic models are followed, in Chapters *15–21*, by some more elaborate models particularly appropriate to life insurance and pensions.

While this book is intended to be comprehensive, it is not meant to be exhaustive. In order to avoid any misunderstanding, we will indicate the limitations of the text:

- Mathematical ideas that could unify and, in some cases, simplify the ideas presented, but which are not included in typical undergraduate courses, are not used. For example, moment generating functions, but not characteristic functions, are used in developments regarding probability distributions. Stieltjes integrals, which could be used in some cases to unify the presentation

of discrete and continuous cases, are not used because of this basic decision on mathematical prerequisites.

- The chapters devoted to life insurance stress the randomness of the time at which a claim payment must be made. In the same chapters, the interest rates used to convert future payments to a present value are considered deterministic and are usually taken as constants. In view of the high volatility possible in interest rates, it is natural to ask why probability models for interest rates were not incorporated. Our answer is that the mathematics of life contingencies on a probabilistic foundation (except for interest) does not involve ideas beyond those covered in an undergraduate program. On the other hand, the modeling of interest rates requires ideas from economics and statistics that are not included in the prerequisites of this volume. In addition, there are some technical problems in building models to combine random interest and random time of claim that are in the process of being solved.
- Methods for estimating the parameters of basic actuarial models from observations are not covered. For example, the construction of life tables is not discussed.
- This is not a text on computing. The issues involved in optimizing the organization of input data and computation in actuarial models are not discussed. This is a rapidly changing area, seemingly best left for readers to resolve as they choose in light of their own resources.
- Many important actuarial problems created by long-term practice and insurance regulation are not discussed. This is true in sections treating topics such as premiums actually charged for life insurance policies, costs reported for pensions, restrictions on benefit provisions, and financial reporting as required by regulators.
- Ideas that lead to interesting puzzles, but which do not appear in basic actuarial models, are avoided. Average age at death problems for a stationary population do not appear for this reason.

This text has a number of features that distinguish it from previous fine textbooks on life contingencies. A number of these features represent decisions by the authors on material to be included and will be discussed under headings suggestive of the topics involved.

Probability Approach

As indicated earlier, the sharpest break between the approach taken here and that taken in earlier English language textbooks on actuarial mathematics is the much fuller use of a probabilistic approach in the treatment of the mathematics of life contingencies. Actuaries have usually written and spoken of applying probabilities in their models, but their results could be, and often were, obtained by a deterministic rate approach. In this work, the treatment of life contingencies is based on the assumption that time-until-death is a continuous-type random variable. This admits a rich field of random variable concepts such as distribution function, probability density function, expected value, variance, and moment generating function. This approach is timely, based on the availability of high-speed

computers, and is called for, based on the observation that the economic role of life insurance and pensions can be best seen when the random value of time-until-death is stressed. Also, these probability ideas are now part of general education in mathematics, and a fuller realization thereof relates life contingencies to other fields of applied probability, for example, reliability theory in engineering.

Additionally, the deterministic rate approach is described for completeness and is a tool in some developments. However, the results obtained from using a deterministic model usually can be obtained as expected values in a probabilistic model.

Integration with Risk Theory

Risk theory is defined as the study of deviations of financial results from those expected and methods of avoiding inconvenient consequences from such deviations. The probabilistic approach to life contingencies makes it easy to incorporate long-term contracts into risk theory models and, in fact, makes life contingencies only a part, but a very important one, of risk theory. Ruin theory, another important part of risk theory, is included as it provides insight into one source, the insurance claims, of adverse long-term financial deviations. This source is the most unique aspect of models for insurance enterprises.

Utility Theory

This text contains topics on the economics of insurance. The goal is to provide a motivation, based on a normative theory of individual behavior in the face of uncertainty, for the study of insurance models. Although the models used are highly simplified, they lead to insights into the economic role of insurance, and to an appreciation of some of the issues that arise in making insurance decisions.

Consistent Assumptions

The assumption of a uniform distribution of deaths in each year of age is consistently used to evaluate actuarial functions at nonintegral ages. This eliminates some of the anomalies that have been observed when inconsistent assumptions are applied in situations involving high interest rates.

Newton L. Bowers
Hans U. Gerber
James C. Hickman
Donald A. Jones
Cecil J. Nesbitt

Introduction to Second Edition

Actuarial science is not static. In the time since the publication of the first edition of *Actuarial Mathematics,* actuarial science has absorbed additional ideas from economics and the mathematical sciences. At the same time, computing and communications have become cheaper and faster, and this has helped to make feasible more complex actuarial models. During this period the financial risks that modern societies seek to manage have also altered as a result of the globalization of business, technological advances, and political shifts that have changed public policies.

It would be impossible to capture the full effect of all these changes in the revision of a basic textbook. Our objective is more modest, but we hope that it is realistic. This edition is a step in an ongoing process of adaptation designed to keep the fundamentals of actuarial science current with changing realities.

In the second edition, changes in notation and nomenclature appear in almost every section. There are also basic changes from the first edition that should be listed.

1. Commutation functions, a classic tool in actuarial calculations, are not used. This is in response to the declining advantages of these functions in an age when interest rates are often viewed as random variables, or as varying deterministically, and the probability distribution of time until decrement may depend on variables other than attained age. Starting in Chapter 3, exercises that illustrate actuarial calculations using recursion formulas that can be implemented with current software are introduced. It is logically necessary that the challenge of implementing tomorrow's software is left to the reader.
2. Utility theory is no longer confined to the first chapter. Examples are given that illustrate how utility theory can be employed to construct consistent models for premiums and reserves that differ from the conventional model that implicitly depends on linear utility of wealth.
3. In the first edition readers were seldom asked to consider more than the first and second moments of loss random variables. In this edition, following the intellectual path used earlier in physics and statistics, the distribution functions and probability density functions of loss variables are illustrated.
4. The basic material on reserves is now presented in two chapters. This facilitates a more complete development of the theory of reserves for general life insurances with varying premiums and benefits.
5. In recent years considerable actuarial research has been done on joint distributions for several future lifetime random variables where mutual independence is not assumed. This work influences the chapters on multiple life actuarial functions and multiple decrement theory.
6. There are potentially serious estimation and interpretation problems in multiple decrement theory when the random times until decrement for competing causes of decrement are not independent. Those problems are illustrated in the second edition.

7. The applications of multiple decrement theory have been consolidated. No attempt is made to illustrate in this basic textbook the variations in benefit formulas driven by rapid changes in pension practice and regulation.
8. The confluence of new research and computing capabilities has increased the use of recursive formulas in calculating the distribution of total losses derived from risk theory models. This development has influenced Chapter 12.
9. The material on pricing life insurance with death and withdrawal benefits and accounting for life insurance operations has been reorganized. Business and regulatory considerations have been concentrated in one chapter, and the foundations of accounting and provisions for expenses in an earlier chapter. The discussion of regulation has been limited to general issues and options for addressing these issues. No attempt has been made to present a definitive interpretation of regulation for any nation, province, or state.
10. The models for some insurance products that are no longer important in the market have been deleted. Models for new products, such as accelerated benefits for terminal illness or long-term care, are introduced.
11. The final chapter contains a brief introduction to simple models in which interest rates are random variables. In addition, ideas for managing interest rate risk are discussed. It is hoped that this chapter will provide a bridge to recent developments within the intersection of actuarial mathematics and financial economics.

As the project of writing this second edition ends, it is clear that a significant new development is under way. This new endeavor is centered on the creation of general models for managing the risks to individuals and organizations created by uncertain future cash flows when the uncertainty derives from any source. This blending of the actuarial/statistical approach to building models for financial security systems with the approach taken in financial economics is a worthy assignment for the next cohort of actuarial students.

Newton L. Bowers
James C. Hickman
Donald A. Jones

Guide to Study

The reader can consider this text as covering the two branches of risk theory. Individual risk theory views each policy as a unit and allows construction of a model for a group of policies by adding the financial results for the separate policies in the group. Collective risk theory uses a probabilistic model for total claims that avoids the step of adding the results for individual policies. This distinction is sometimes difficult to maintain in practice. The chapters, however, can be classified as illustrated below.

Individual Risk Theory	Collective Risk Theory
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	12, 13, 14, 19, 20

It is also possible to divide insurance models into those appropriate for short-term insurance, where investment income is not a significant factor, and long-term insurance, where investment income is important. The following classification scheme provides this division of chapters along with an additional division of long-term models between those for life insurance and those for pensions.

	Long-Term Insurances	
Short-Term Insurances	Life Insurance	Pensions
1, 2, 12, 13, 14	3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 21	9, 10, 11, 19, 20, 21

The selection of topics and their organization do not follow a traditional pattern. As stated previously, the new organization arose from the goal to first cover material considered basic for all actuarial students (Chapters 1–14) and then to include a more in-depth treatment of selected topics for students specializing in life insurance and pensions (Chapters 15–21).

The discussion in Chapter 1 is devoted to two ideas: that random events can disrupt the plans of decision makers and that insurance systems are designed to reduce the adverse financial effects of these events. To illustrate the latter, single insurance policies are discussed and convenient, if not necessarily realistic, distributions of the loss random variable are used. In subsequent chapters, more detailed models are constructed for use with insurance systems.

In Chapter 2, the individual risk model is developed, first in regard to single policies, then in regard to a portfolio of policies. In this model, a random variable, S, the total claims in a single period, is the sum of a fixed number of independent random variables, each of which is associated with a single policy. Each component of the sum S can take either the value 0 or a random claim amount in the course of a single period.

From the viewpoint of risk theory, the ideas developed in Chapters 3 through 11 can be seen as extending the ideas of Chapter 2. Instead of considering the potential claims in a short period from an individual policy, we consider loss variables that take into account the financial results of several periods. Since such random variables are no longer restricted to a short time period, they reflect the time value of money. For groups of individuals, we can then proceed, as in

Chapter 2, to use an approximation, such as the normal approximation, to make probability statements about the sum of the random variables that are associated with the individual members.

In Chapter 3, time-of-death is treated as a continuous random variable, and, after defining the probability density function, several features of the probability distribution are introduced and explored. In Chapters 4 and 5, life insurances and annuities are introduced, and the present values of the benefits are expressed as functions of the time-of-death. Several characteristics of the distributions of the present value of future benefits are examined. In Chapter 6, the equivalence principle is introduced and used to define and evaluate periodic benefit premiums. In Chapters 7 and 8, the prospective future loss on a contract already in force is investigated. The distribution of future loss is examined, and the benefit reserve is defined as the expected value of this loss. In Chapter 9, annuity and insurance contracts involving two lives are studied. (Discussion of more advanced multiple life theory is deferred until Chapter 18.) The discussion in Chapters 10 and 11 investigates a more realistic model in which several causes of decrement are possible. In Chapter 10, basic theory is examined, whereas in Chapter 11 the theory is applied to calculating actuarial present values for a variety of insurance and pension benefits.

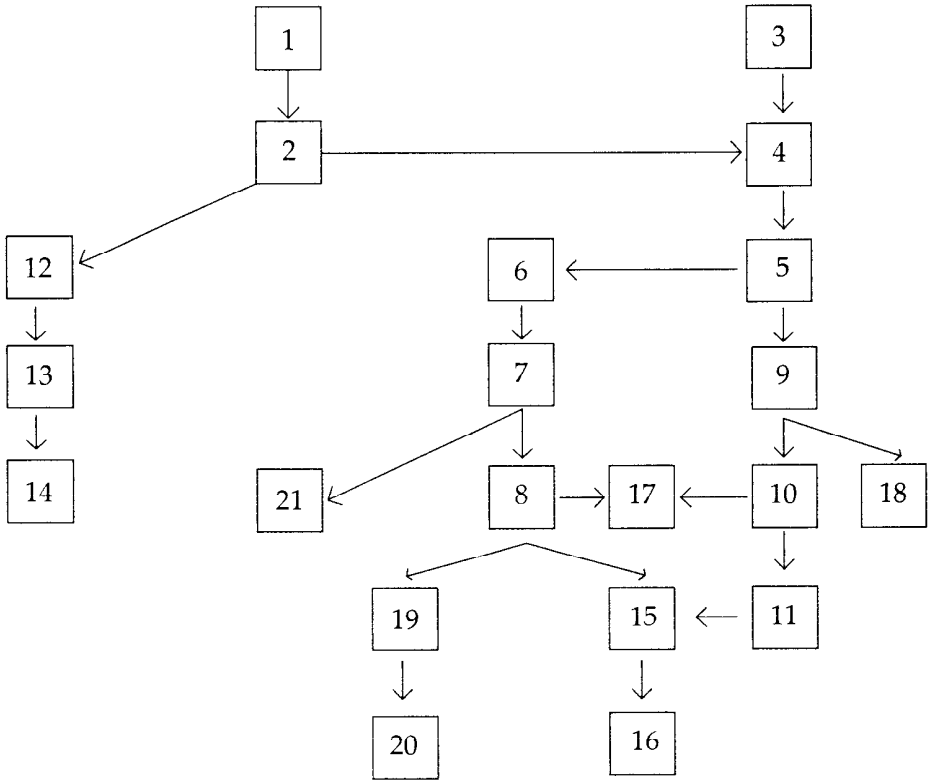
In Chapter 12, the collective risk model is developed with respect to single-period considerations of a portfolio of policies. The distribution of total claims for the period is developed by postulating the characteristics of the portfolio in the aggregate rather than as a sum of individual policies. In Chapter 13, these ideas are extended to a continuous-time model that can be used to study solvency requirements over a long time period. Applications of risk theory to insurance models are given an overview in Chapter 14.

Elaboration of the individual model to incorporate operational constraints such as acquisition and administrative expenses, accounting requirements, and the effects of contract terminations is treated in Chapters 15 and 16. In Chapter 17, individual risk theory models are used to obtain actuarial present values, benefit and contract premiums, and benefit reserves for selected special plans including life annuities with certain periods that depend on the contract premium, variable and flexible products, and accelerated benefits. In Chapter 18, the elementary models for plans involving two lives are extended to incorporate contingencies based on a larger number of lives and more complicated benefits.

In Chapter 19, concepts of population theory are introduced. These concepts are then applied to tracing the progress of life insurance benefits provided on a group, or population, basis. The tools from population theory are applied to tracing the progress of retirement income benefits provided on a group basis in Chapter 20.

Chapter 21 is a step into the future. Interest rates are assumed to be random variables. Several stochastic models are introduced and then integrated into models for basic insurance and annuity contracts.

The following diagram illustrates the prerequisite structure of the chapters. The arrows indicate the direction of the flow. For any chapter, the chapters that are upstream are prerequisite. For example, Chapter 6 has as prerequisites Chapters 1, 2, 3, 4, and 5.



We have a couple of hints for the reader, particularly for one for whom the material is new. The exercises are an important part of the text and include material not covered in the main discussion. In some cases, hints will be offered to aid in the solution. Answers to all exercises are provided except where the answer is given in the formulation of the problem. Writing computer programs and using electronic spreadsheets or mathematical software for the evaluation of basic formulas are excellent ways of enhancing the level of understanding of the material. The student is encouraged to use these tools to work through the computing exercises.

We conclude these introductory comments with some miscellaneous information on the format of the text. First, each chapter concludes with a reference section that provides guidance to those who wish to pursue further study of the topics covered in the chapter. These sections also contain comments that relate the ideas used in insurance models to those used in other areas.

Second, Chapters 1, 12, 13, 14, and 18 contain some theorems with their proofs included as chapter appendices. These proofs are included for completeness, but

are not essential to an understanding of the material. They may be excluded from study at the reader's discretion. Exercises associated with these appendices should also be considered optional.

Third, general appendices appear at the end of the text. Included here are numerical tables for computations for examples and exercises, an index to notation, a discussion of general rules for writing actuarial symbols, reference citations, answers to exercises, a subject index, and supplemental mathematical formulas that are not assumed to be a part of the mathematical prerequisites.

Fourth, we observe two notational conventions. A referenced random variable, X, for example, is designated with a capital letter. This notational convention is not used in older texts on probability theory. It will be our practice, in order to indicate the correspondence, to use the appropriate random variable symbol as a subscript on functions and operators that depend on the random variable. We will use the general abbreviation *log* to refer to natural (base e) logarithms, because a distinction between natural and common logarithms is unnecessary in the examples and exercises. We assume the natural logarithm in our computations.

Fifth, currencies such as dollar, pound, lira, or yen are not specified in the examples and exercises due to the international character of the required computations.

Finally, since we have discussed prerequisites to this work, some major theorems from undergraduate calculus and probability theory will be used without review or restatement in the discussions and exercises.



THE ECONOMICS OF INSURANCE

1.1 Introduction

Each of us makes plans and has expectations about the path his or her life will follow. However, experience teaches that plans will not unfold with certainty and sometimes expectations will not be realized. Occasionally plans are frustrated because they are built on unrealistic assumptions. In other situations, fortuitous circumstances interfere. Insurance is designed to protect against serious financial reversals that result from random events intruding on the plans of individuals.

We should understand certain basic limitations on insurance protection. First, it is restricted to reducing those consequences of random events that can be measured in monetary terms. Other types of losses may be important, but not amenable to reduction through insurance.

For example, pain and suffering may be caused by a random event. However, insurance coverages designed to compensate for pain and suffering often have been troubled by the difficulty of measuring the loss in monetary units. On the other hand, economic losses can be caused by events such as property set on fire by its owner. Whereas the monetary terms of such losses may be easy to define, the events are not insurable because of the nonrandom nature of creating the losses.

A second basic limitation is that insurance does not directly reduce the probability of loss. The existence of windstorm insurance will not alter the probability of a destructive storm. However, a well-designed insurance system often provides financial incentives for loss prevention activities. An insurance product that encouraged the destruction of property or the withdrawal of a productive person from the labor force would affect the probability of these economically adverse events. Such insurance would not be in the public interest.

Several examples of situations where random events may cause financial losses are the following:

- The destruction of property by fire or storm is usually considered a random event in which the loss can be measured in monetary terms.

- A damage award imposed by a court as a result of a negligent act is often considered a random event with resulting monetary loss.
- Prolonged illness may strike at an unexpected time and result in financial losses. These losses will be due to extra health care expenses and reduced earned income.
- The death of a young adult may occur while long-term commitments to family or business remain unfulfilled. Or, if the individual survives to an advanced age, resources for meeting the costs of living may be depleted.

These examples are designed to illustrate the definition:

> An ***insurance system*** is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations.

It is helpful to make certain distinctions between insurance and related systems. Banking institutions were developed for the purpose of receiving, investing, and dispensing the savings of individuals and corporations. The cash flows in and out of a savings institution do not follow deterministic paths. However, unlike insurance systems, savings institutions do not make payments based on the size of a financial loss occurring from an event outside the control of the person suffering the loss.

Another system that does make payments based on the occurrence of random events is gambling. Gambling or wagering, however, stands in contrast to an insurance system in that an insurance system is designed to protect against the economic impact of risks that exist independently of, and are largely beyond the control of, the insured. The typical gambling arrangement is established by defining payoff rules about the occurrence of a contrived event, and the risk is voluntarily sought by the participants. Like insurance, a gambling arrangement typically redistributes wealth, but it is there that the similarity ends.

Our definition of an insurance system is purposefully broad. It encompasses systems that cover losses in both property and human-life values. It is intended to cover insurance systems based on individual decisions to participate as well as systems where participation is a condition of employment or residence. These ideas are discussed in Section 1.4.

The economic justification for an insurance system is that it contributes to general welfare by improving the prospect that plans will not be frustrated by random events. Such systems may also increase total production by encouraging individuals and corporations to embark on ventures where the possibility of large losses would inhibit such projects in the absence of insurance. The development of marine insurance, for reducing the financial impact of the perils of the sea, is an example of this point. Foreign trade permitted specialization and more efficient production, yet mutually advantageous trading activity might be too hazardous for some potential trading partners without an insurance system to cover possible losses at sea.

1.2 Utility Theory

If people could foretell the consequences of their decisions, their lives would be simpler but less interesting. We would all make decisions on the basis of preferences for certain consequences. However, we do not possess perfect foresight. At best, we can select an action that will lead to one set of uncertainties rather than another. An elaborate theory has been developed that provides insights into decision making in the face of uncertainty. This body of knowledge is called ***utility theory***. Because of its relevance to insurance systems, its main points will be outlined here.

One solution to the problem of decision making in the face of uncertainty is to define the value of an economic project with a random outcome to be its expected value. By this ***expected value principle*** the distribution of possible outcomes may be replaced for decision purposes by a single number, the expected value of the random monetary outcomes. By this principle, a decision maker would be indifferent between assuming the random loss X and paying amount $E[X]$ in order to be relieved of the possible loss. Similarly, a decision maker would be willing to pay up to $E[Y]$ to participate in a gamble with random payoff Y. In economics the expected value of random prospects with monetary payments is frequently called the ***fair*** or ***actuarial value*** of the prospect.

Many decision makers do not adopt the expected value principle. For them, their wealth level and other aspects of the distribution of outcomes influence their decisions.

Below is an illustration designed to show the inadequacy of the expected value principle for a decision maker considering the value of accident insurance. In all cases, it is assumed that the probability of an accident is 0.01 and the probability of no accident is 0.99. Three cases are considered according to the amount of loss arising from an accident; the expected loss is tabulated for each.

Case	Possible Losses	Expected Loss
1	0	1	0.01
2	0	1 000	10.00
3	0	1 000 000	10 000.00

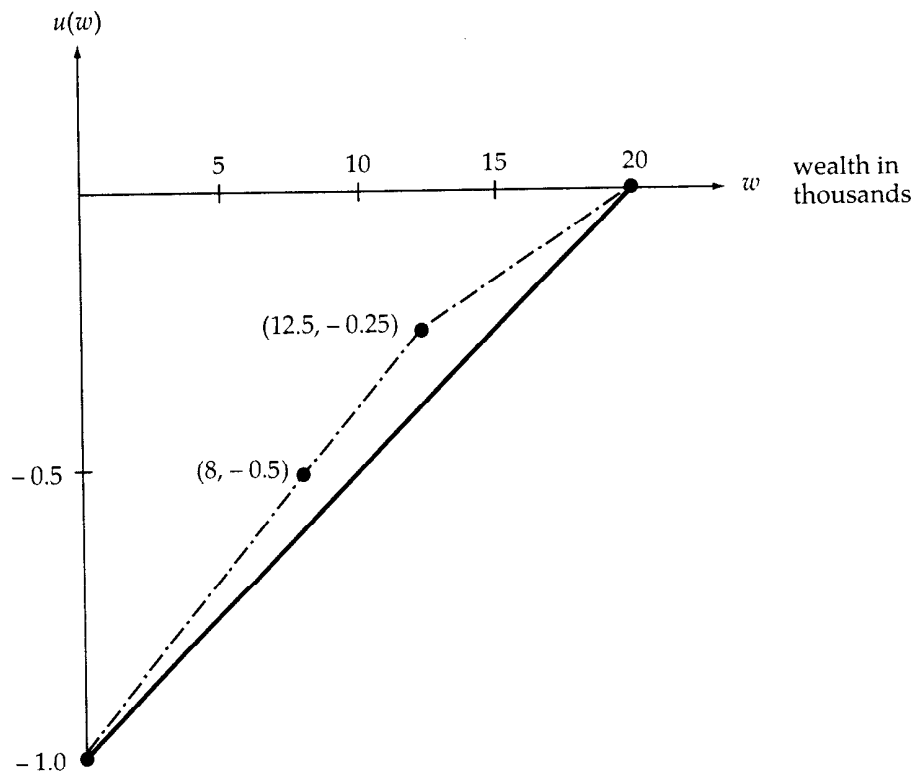
A loss of 1 might be of little concern to the decision maker who then might be unwilling to pay more than the expected loss to obtain insurance. However, the loss of 1,000,000, which may exceed his net worth, could be catastrophic. In this case, the decision maker might well be willing to pay more than the expected loss of 10,000 in order to obtain insurance. The fact that the amount a decision maker would pay for protection against a random loss may differ from the expected value suggests that the expected value principle is inadequate to model behavior.

We now study another approach to explain why a decision maker may be willing to pay more than the expected value. At first we simply assume that the value or utility that a particular decision maker attaches to wealth of amount w, measured in monetary units, can be specified in the form of a function $u(w)$, called a ***utility function.*** We demonstrate a procedure by which a few values of such a function can be determined. For this we assume that our decision maker has wealth equal to 20,000. A linear transformation,

$$u^*(w) = a\,u(w) + b \qquad a > 0,$$

yields a function $u^*(w)$, which is essentially equivalent to $u(w)$. It then follows by choice of a and b that we can determine arbitrarily the 0 point and one additional point of an individual's utility function. Therefore, we fix $u(0) = -1$ and $u(20{,}000) = 0$. These values are plotted on the solid line in Figure 1.2.1.

Determination of a Utility Function



We now ask a question of our decision maker: Suppose you face a loss of 20,000 with probability 0.5, and will remain at your current level of wealth with probability 0.5. What is the maximum amount* G you would be willing to pay for

*Premium quantities, by convention in insurance literature, are capitalized although they are not random variables.

complete insurance protection against this random loss? We can express this question in the following way: For what value of G does

$$u(20{,}000 - G) = 0.5\, u(20{,}000) + 0.5\, u(0)$$
$$= (0.5)(0) + (0.5)(-1) = -0.5?$$

If he pays amount G, his wealth will certainly remain at $20{,}000 - G$. The equal sign indicates that the decision maker is indifferent between paying G with certainty and accepting the expected utility of wealth expressed on the right-hand side.

Suppose the decision maker's answer is $G = 12{,}000$. Therefore,

$$u(20{,}000 - 12{,}000) = u(8{,}000) = -0.5.$$

This result is plotted on the dashed line in Figure 1.2.1. Perhaps the most important aspect of the decision maker's response is that he is willing to pay an amount for insurance that is greater than

$$(0.5)(0) + (0.5)(20{,}000) = 10{,}000,$$

the expected value of the loss.

This procedure can be used to add as many points $[w, u(w)]$, for $0 \le w \le 20{,}000$, as needed to obtain a satisfactory approximation to the decision maker's utility of wealth function. Once a utility value has been assigned to wealth levels w_1 and w_2, where $0 \le w_1 < w_2 \le 20{,}000$, we can determine an additional point by asking the decision maker the following question: What is the maximum amount you would pay for complete insurance against a situation that could leave you with wealth w_2 with specified probability p, or at reduced wealth level w_1 with probability $1 - p$? We are asking the decision maker to fix a value G such that

$$u(w_2 - G) = (1 - p)u(w_1) + p\, u(w_2). \tag{1.2.1}$$

Once the value $w_2 - G = w_3$ is available, the point $[w_3, (1 - p)u(w_1) + p\, u(w_2)]$ is determined as another point of the utility function. Such a process has been used to assign a fourth point (12,500, −0.25) in Figure 1.2.1. Such solicitation of preferences leads to a set of points on the decision maker's utility function. A smooth function with a second derivative may be fitted to these points to provide for a utility function everywhere.

After a decision maker has determined his utility of wealth function by the method outlined, the function can be used to compare two random economic prospects. The prospects are denoted by the random variables X and Y. We seek a decision rule that is consistent with the preferences already elicited in the determination of the utility of wealth function. Thus, if the decision maker has wealth w, and must compare the random prospects X and Y, the decision maker selects X if

$$\mathrm{E}[u(w + X)] > \mathrm{E}[u(w + Y)],$$

and the decision maker is indifferent between X and Y if

$$E[u(w + X)] = E[u(w + Y)].$$

Although the method of eliciting and using a utility function may seem plausible, it is clear that our informal development must be augmented by a more rigorous chain of reasoning if utility theory is to provide a coherent and comprehensive framework for decision making in the face of uncertainty. If we are to understand the economic role of insurance, such a framework is needed. An outline of this more rigorous theory follows.

The theory starts with the assumption that a rational decision maker, when faced with two distributions of outcomes affecting wealth, is able to express a preference for one of the distributions or indifference between them. Furthermore, the preferences must satisfy certain consistency requirements. The theory culminates in a theorem stating that if preferences satisfy the consistency requirements, there is a utility function $u(w)$ such that if the distribution of X is preferred to the distribution of Y, $E[u(X)] > E[u(Y)]$, and if the decision maker is indifferent between the two distributions, $E[u(X)] = E[u(Y)]$. That is, the qualitative preference or indifference relation may be replaced by a consistent numerical comparison. In Section 1.6, references are given for the detailed development of this theory.

Before turning to applications of utility theory for insights into insurance, we record some observations about utility.

Observations:

1. Utility theory is built on the assumed existence and consistency of preferences for probability distributions of outcomes. A utility function should reveal no surprises. It is a numerical description of existing preferences.
2. A utility function need not, in fact, cannot, be determined uniquely. For example, if

   $$u^*(w) = a\,u(w) + b \qquad a > 0,$$

   then

   $$E[u(X)] > E[u(Y)]$$

   is equivalent to

   $$E[u^*(X)] > E[u^*(Y)].$$

   That is, preferences are preserved when the utility function is an increasing linear transformation of the original form. This fact was used in the Figure 1.2.1 illustration where two points were chosen arbitrarily.
3. Suppose the utility function is linear with a positive slope; that is,

   $$u(w) = a\,w + b \qquad a > 0.$$

   Then, if $E[X] = \mu_X$ and $E[Y] = \mu_Y$, we have

   $$E[u(X)] = a\,\mu_X + b > E[u(Y)] = a\,\mu_Y + b$$

   if and only if $\mu_X > \mu_Y$. That is, for increasing linear utility functions, preferences

for distributions of outcomes are in the same order as the expected values of the distributions being compared. Therefore, the expected value principle for rational economic behavior in the face of uncertainty is consistent with the expected utility rule when the utility function is an increasing linear one.

1.3 Insurance and Utility

In Section 1.2 we outlined utility theory for the purpose of gaining insights into the economic role of insurance. To examine this role we start with an illustration. Suppose a decision maker owns a property that may be damaged or destroyed in the next accounting period. The amount of the loss, which may be 0, is a random variable denoted by X. We assume that the distribution of X is known. Then $E[X]$, the expected loss in the next period, may be interpreted as the long-term average loss if the experiment of exposing the property to damage may be observed under identical conditions a great many times. It is clear that this long-term set of trials could not be performed by an individual decision maker.

Suppose that an insurance organization (***insurer***) was established to help reduce the financial consequences of the damage or destruction of property. The insurer would issue contracts (***policies***) that would promise to pay the owner of a property a defined amount equal to or less than the financial loss if the property were damaged or destroyed during the period of the policy. The contingent payment linked to the amount of the loss is called a ***claim*** payment. In return for the promise contained in the policy, the owner of the property (***insured***) pays a consideration (***premium***).

The amount of the premium payment is determined after an economic decision principle has been adopted by each of the insurer and insured. An opportunity exists for a mutually advantageous insurance policy when the premium for the policy set by the insurer is less than the maximum amount that the property owner is willing to pay for insurance.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this case, the insurer would adopt the expected value principle in setting its premium, as indicated in Section 1.2, Observation 3; that is, the insurer would set its basic price for full insurance coverage as the expected loss, $E[X] = \mu$. In this context μ is called the ***pure*** or ***net premium*** for the 1-period insurance policy. To provide for expenses, taxes, and profit and for some security against adverse loss experience, the insurance system would decide to set the premium for the policy by ***loading,*** adding to, the pure premium. For instance, the loaded premium, denoted by H, might be given by

$$H = (1 + a)\mu + c \qquad a > 0, \qquad c > 0.$$

In this expression the quantity $a\mu$ can be viewed as being associated with expenses that vary with expected losses and with the risk that claims experience will deviate from expected. The constant c provides for expected expenses that do not vary with

losses. Later, we will illustrate other economic principles for determining premiums that might be adopted by the insurer.

We now apply utility theory to the decision problems faced by the owner of the property subject to loss. The property owner has a utility of wealth function $u(w)$ where wealth w is measured in monetary terms. The owner faces a possible loss due to random events that may damage the property. The distribution of the random loss X is assumed to be known. Much as in (1.2.1), the owner will be indifferent between paying an amount G to the insurer, who will assume the random financial loss, and assuming the risk himself. This situation can be stated as

$$u(w - G) = \mathrm{E}[u(w - \mathrm{X})]. \tag{1.3.1}$$

The right-hand side of (1.3.1) represents the expected utility of not buying insurance when the owner's current wealth is w. The left-hand side of (1.3.1) represents the expected utility of paying G for complete financial protection.

If the owner has an increasing linear utility function, that is, $u(w) = bw + d$ with $b > 0$, the owner will be adopting the expected value principle. In this case the owner prefers, or is indifferent to, the insurance when

$$u(w - G) = b(w - G) + d \geq \mathrm{E}[u(w - \mathrm{X})] = \mathrm{E}[b(w - \mathrm{X}) + d],$$

$$b(w - G) + d \geq b(w - \mu) + d,$$

$$G \leq \mu.$$

That is, if the owner has an increasing linear utility function, the premium payments that will make the owner prefer, or be indifferent to, complete insurance are less than or equal to the expected loss. In the absence of a subsidy, an insurer, over the long term, must charge more than its expected losses. Therefore, in this case, there seems to be little opportunity for a mutually advantageous insurance contract. If an insurance contract is to result, the insurer must charge a premium in excess of expected losses and expenses to avoid a bias toward insufficient income. The property owner then cannot use a linear utility function.

In Section 1.2 we mention that the preferences of a decision maker must satisfy certain consistency requirements to ensure the existence of a utility function. Although these requirements were not listed, they do not include any specifications that would force a utility function to be linear, quadratic, exponential, logarithmic, or any other particular form. In fact, each of these named functions might serve as a utility function for some decision maker or they might be spliced together to reflect some other decision maker's preferences.

Nevertheless, it seems natural to assume that $u(w)$ is an increasing function, "more is better." In addition, it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is the idea of decreasing marginal utility in economics.

The approximate utility function of Figure 1.2.1 consists of straight line segments with positive slopes. It is such that $\Delta^2 u(w) \leq 0$. If these ideas are extended to

smoother functions, the two properties suggested by observation are $u'(w) > 0$ and $u''(w) < 0$. The second inequality indicates that $u(w)$ is a strictly concave downward function.

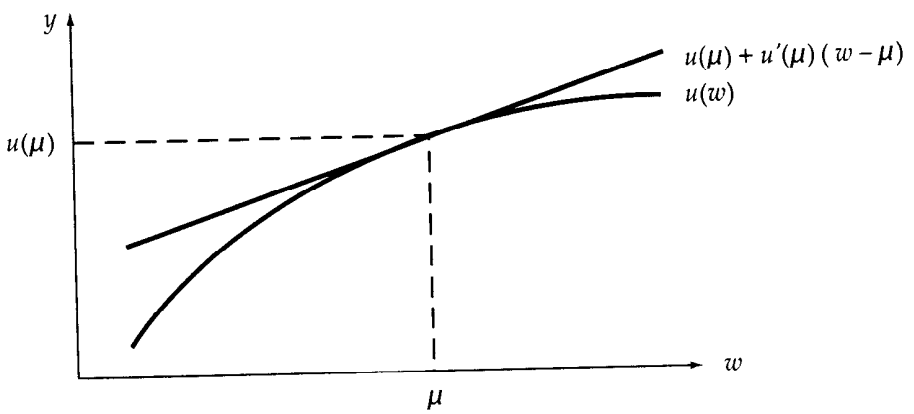
In discussing insurance decisions using strictly concave downward utility functions, we will make use of one form of ***Jensen's inequalities.*** These inequalities state that for a random variable X and function $u(w)$,

$$\text{if } u''(w) < 0, \text{ then } \mathrm{E}[u(X)] \leq u(\mathrm{E}[X]), \tag{1.3.2}$$

$$\text{if } u''(w) > 0, \text{ then } \mathrm{E}[u(X)] \geq u(\mathrm{E}[X]). \tag{1.3.3}$$

Jensen's inequalities require the existence of the two expected values. Proofs of the inequalities are required by Exercise 1.3. A second proof of (1.3.2) is almost immediate from consideration of Figure 1.3.1 as follows.

Proof of Jensen's Inequalities for the Case $u'(w) > 0$ and $u''(w) < 0$



If $\mathrm{E}[X] = \mu$ exists, one considers the line tangent to $u(w)$,

$$y = u(\mu) + u'(\mu)(w - \mu),$$

at the point $(\mu, u(\mu))$. Because of the strictly concave characteristic of $u(w)$, the graph of $u(w)$ will be below the tangent line; that is,

$$u(w) \leq u(\mu) + u'(\mu)(w - \mu) \tag{1.3.4}$$

for all values of w. If w is replaced by the random variable X, and the expectation is taken on each side of the inequality (1.3.4), we have $\mathrm{E}[u(X)] \leq u(\mu)$.

This basic inequality has several applications in actuarial mathematics. Let us apply Jensen's inequality (1.3.2) to the decision maker's insurance problem as formulated in (1.3.1). We will assume that the decision maker's preferences are such that $u'(w) > 0$ and $u''(w) < 0$. Applying Jensen's inequality to (1.3.1), we have

$$u(w - G) = \mathrm{E}[u(w - X)] \leq u(w - \mu). \tag{1.3.5}$$

Because $u'(w) > 0$, $u(w)$ is an increasing function. Therefore, (1.3.5) implies that $w - G \leq w - \mu$, or $G \geq \mu$ with $G > \mu$ unless X is a constant. In economic terms, we have found that if $u'(w) > 0$ and $u''(w) < 0$, the decision maker will pay an amount greater than the expected loss for insurance. If G is at least equal to the premium set by the insurer, there is an opportunity for a mutually advantageous insurance policy.

Formally we say a decision maker with utility function $u(w)$ is ***risk averse*** if, and only if, $u''(w) < 0$.

We now employ a general utility function for the insurer. We let $u_I(w)$ denote the utility of wealth function of the insurer and w_I denote the current wealth of the insurer measured in monetary terms. Then the minimum acceptable premium H for assuming random loss X, from the viewpoint of the insurer, may be determined from (1.3.6):

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)]. \tag{1.3.6}$$

The left-hand side of (1.3.6) is the utility attached to the insurer's current position. The right-hand side is the expected utility associated with collecting premium H and paying random loss X. In other words, the insurer is indifferent between the current position and providing insurance for X at premium H. If the insurer's utility function is such that $u_I'(w) > 0$, $u_I''(w) < 0$, we can use Jensen's inequality (1.3.2) along with (1.3.6) to obtain

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)] \leq u_I(w_I + H - \mu).$$

Following the same line of reasoning displayed in connection with (1.3.5), we can conclude that $H \geq \mu$. If G, as determined by the decision maker by solving (1.3.5), is such that $G \geq H \geq \mu$, an insurance contract is possible. That is, the expected utility of neither party to the contract is decreased.

A utility function is based on the decision maker's preferences for various distributions of outcomes. An insurer need not be an individual. It may be a partnership, corporation, or government agency. In this situation the determination of $u_I(w)$, the insurer's utility function, may be a rather complicated matter. For example, if the insurer is a corporation, one of management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflicting attitudes toward risk among the groups of stockholders.

Several elementary functions are used to illustrate properties of utility functions. Here we examine exponential, fractional power, and quadratic functions. Exercises 1.6, 1.8, 1.9, 1.10(b), and 1.13 cover the logarithmic utility function.

An ***exponential utility function*** is of the form

$$u(w) = -e^{-\alpha w} \qquad \text{for all } w \text{ and for a fixed } \alpha > 0$$

and has several attractive features. First,

$$u'(w) = \alpha e^{-\alpha w} > 0.$$

Second,

$$u''(w) = -\alpha^2 e^{-\alpha w} < 0.$$

Therefore, $u(w)$ may serve as the utility function of a risk-averse individual. Third, finding

$$E[-e^{-\alpha X}] = -E[e^{-\alpha X}] = -M_X(-\alpha)$$

is essentially the same as finding the moment generating function (m.g.f.) of X. In this expression,

$$M_X(t) = E[e^{tX}]$$

denotes the m.g.f. of X. Fourth, insurance premiums do not depend on the wealth of the decision maker. This statement is verified for the insured by substituting the exponential utility function into (1.3.1). That is,

$$-e^{-\alpha(w-G)} = E[-e^{-\alpha(w-X)}],$$

$$e^{\alpha G} = M_X(\alpha),$$

$$G = \frac{\log M_X(\alpha)}{\alpha}$$

and G does not depend on w.

The verification for the insurer is done by substituting the exponential utility function with parameter α_I into (1.3.6):

$$-e^{-\alpha_I w_I} = E[-e^{-\alpha_I(w_I + H - X)}],$$

$$-e^{-\alpha_I w_I} = -e^{-\alpha_I(w_I + H)} M_X(\alpha_I),$$

$$H = \frac{\log M_X(\alpha_I)}{\alpha_I}.$$

Example 1.3.1

A decision maker's utility function is given by $u(w) = -e^{-5w}$. The decision maker has two random economic prospects (gains) available. The outcome of the first, denoted by X, has a normal distribution with mean 5 and variance 2. Henceforth, a statement about a normal distribution with mean μ and variance σ^2 will be abbreviated as $N(\mu, \sigma^2)$. The second prospect, denoted by Y, is distributed as $N(6, 2.5)$. Which prospect will be preferred?

Solution:

We have

$$\begin{aligned} E[u(X)] &= E[-e^{-5X}] \\ &= -M_X(-5) = -e^{[-5(5)+(5^2)(2)/2]} \\ &= -1, \end{aligned}$$

and

$$\begin{aligned} E[u(Y)] &= E[-e^{-5Y}] \\ &= -M_Y(-5) = -e^{[-5(6)+(5^2)(2.5)/2]} \\ &= -e^{1.25}. \end{aligned}$$

Therefore,

$$E[u(X)] = -1 > E[u(Y)] = -e^{1.25},$$

and the distribution of X is preferred to the distribution of Y. ▼

In Example 1.3.1 prospect X is preferred to Y despite the fact that $\mu_X = 5 < \mu_Y = 6$. Since the decision maker is risk averse, the fact that the distribution of Y is more diffuse than the distribution of X is weighted heavily against the distribution of Y in assessing its desirability. If Y had a $N(6, 2.4)$ distribution, $E[u(Y)] = -1$ and the decision maker would be indifferent between the distributions of X and Y.

The family of ***fractional power utility*** functions is given by

$$u(w) = w^\gamma \qquad w > 0,\ 0 < \gamma < 1.$$

A member of this family might represent the preferences of a risk-averse decision maker since

$$u'(w) = \gamma w^{\gamma-1} > 0$$

and

$$u''(w) = \gamma(\gamma - 1)w^{\gamma-2} < 0.$$

In this family, premiums depend on the wealth of the decision maker in a manner that may be sufficiently realistic in many situations.

Example 1.3.2

A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of $w = 10$ and faces a random loss X with a uniform distribution on $(0, 10)$. What is the maximum amount this decision maker will pay for complete insurance against the random loss?

Solution:

Substituting into (1.3.1) we have

$$\begin{aligned} \sqrt{10 - G} &= E[\sqrt{10 - X}] \\ &= \int_0^{10} \sqrt{10 - x}\ 10^{-1}\, dx \\ &= \frac{-2(10 - x)^{3/2}}{3(10)}\Bigg|_0^{10} \\ &= \frac{2}{3}\sqrt{10}, \\ G &= 5.5556. \end{aligned}$$

The decision maker is risk averse and has $u'(w) > 0$. Following the discussion of (1.3.5), we would expect $G > E[X]$, and in this example $G = 5.5556 > E[X] = 5$.▼

The family of ***quadratic utility*** functions is given by

$$u(w) = w - \alpha w^2 \qquad w < (2\alpha)^{-1}, \qquad \alpha > 0.$$

A member of this family might represent the preferences of a risk-averse decision maker since $u''(w) = -2\alpha$. While a quadratic utility function is convenient because decisions depend only on the first two moments of the distributions of outcomes under consideration, there are certain consequences of its use that strike some people as being unreasonable. Example 1.3.3 illustrates one of these consequences.

Example 1.3.3

A decision maker's utility of wealth function is given by

$$u(w) = w - 0.01w^2 \qquad w < 50.$$

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount c with probability $1 - p$. For the values of w, c, and p exhibited in the table below, find the maximum insurance premium that the decision maker will pay for complete insurance. Assume $c \leq w < 50$.

Solution:

For the facts stated, (1.3.1) becomes

$$u(w - G) = pu(w) + (1 - p)u(w - c),$$

$$(w - G) - 0.01(w - G)^2 = p(w - 0.01w^2) + (1 - p)[(w - c) - 0.01(w - c)^2].$$

For given values of w, p, and c this expression becomes a quadratic equation. Two solutions are shown.

Wealth w	Loss c	Probability p	Insurance Premium G
10	10	0.5	5.28
20	10	0.5	5.37

▼

In Example 1.3.3, as anticipated, G is greater than the expected loss of 5. However, the maximum insurance premium for exactly the same loss distribution increases with the wealth of the decision maker. This result seems unreasonable to some who anticipate that more typical behavior would be a decrease in the amount a decision maker would pay for insurance when an increase in wealth would permit the decision maker to absorb more of a random loss. Unfortunately, a maximum insurance premium that increases with wealth is a property of quadratic utility functions. Consequently, these utility functions should not be selected by a decision maker who perceives that his ability to absorb random losses goes up with increases in wealth.

If we rework Example 1.3.3 using an exponential utility function, we know that the premium G will not depend on w, the amount of wealth. In fact, if $u(w) = -e^{-0.01w}$, it can be shown that $G = 5.12$ for both $w = 10$ and $w = 20$.

Example 1.3.4

The probability that a property will not be damaged in the next period is 0.75. The probability density function (p.d.f.) of a positive loss is given by

$$f(x) = 0.25(0.01e^{-0.01x}) \qquad x > 0.$$

The owner of the property has a utility function given by

$$u(w) = -e^{-0.005w}.$$

Calculate the expected loss and the maximum insurance premium the property owner will pay for complete insurance.

Solution:

The expected loss is given by

$$\begin{aligned} \mathrm{E}[X] &= 0.75(0) + 0.25 \int_0^\infty x(0.01e^{-0.01x})dx \\ &= 25. \end{aligned}$$

We apply (1.3.1) to determine the maximum premium that the owner will pay for complete insurance. This premium will be consistent with the property owner's preferences as summarized in the utility function:

$$u(w - G) = 0.75u(w) + \int_0^\infty u(w - x)f(x)dx,$$

$$-e^{-0.005(w-G)} = -0.75e^{-0.005w} - 0.25 \int_0^\infty e^{-0.005(w-x)}(0.01e^{-0.01x})dx,$$

$$\begin{aligned} e^{0.005G} &= 0.75 + (0.25)(2) \\ &= 1.25, \\ G &= 200 \log 1.25 \\ &= 44.63. \end{aligned}$$

Therefore, in accord with the property owner's preferences, he will pay up to $44.63 - 25 = 19.63$ in excess of the expected loss to purchase insurance covering all losses in the next period. ▼

In Example 1.3.5 the notion of insurance that covers something less than the complete loss is introduced. A modification is made in (1.3.1) to accommodate the fact that losses are shared by the decision maker and the insurance system.

Example 1.3.5

The property owner in Example 1.3.4 is offered an insurance policy that will pay 1/2 of any loss during the next period. The expected value of the partial loss

payment is $E[X/2] = 12.50$. Calculate the maximum premium that the property owner will pay for this insurance.

Solution:

Consistent with his attitude toward risk, as summarized in his utility function, the premium is determined from

$$0.75u(w - G) + \int_0^\infty u\left(w - G - \frac{x}{2}\right) f(x)dx$$

$$= 0.75u(w) + \int_0^\infty u(w - x)f(x)dx.$$

The left-hand side of this equation represents the expected utility with the partial insurance coverage. The right-hand side represents the expected utility with no insurance. For the exponential utility function and p.d.f. of losses specified in Example 1.3.4, it can be shown that $G = 28.62$. The property owner is willing to pay up to $G - \mu = 28.62 - 12.50 = 16.12$ more than the expected partial loss for the partial insurance coverage. ▼

1.4 Elements of Insurance

Individuals and organizations face the threat of financial loss due to random events. In Section 1.3 we saw how insurance can increase the expected utility of a decision maker facing such random losses. Insurance systems are unique in that the alleviation of financial losses in which the number, size, or time of occurrence is random is the primary reason for their existence. In this section we review some of the factors influencing the organization and management of an insurance system.

An insurance system can be organized only after the identification of a class of situations where random losses may occur. The word random is taken to mean, along with other attributes, that the frequency, size, or time of loss is not under the control of the prospective insured. If such control exists, or if a claim payment exceeds the actual financial loss, an incentive to incur a loss will exist. In such a situation, the assumptions under which the insurance system was organized will become invalid. The actual conditions under which premiums are collected and claims paid will be different from those assumed in organizing the system. The system will not achieve its intended objective of not decreasing the expected utilities of both the insured and the insurer.

Once a class of insurable situations is identified, information on the expected utilities and the loss-generating process can be obtained. Market research in insurance can be viewed as an effort to learn about the utility functions, that is, the risk preferences of consumers.

The processes generating size and time of loss may be sufficiently stable over time so that past information can be used to plan the system. When a new insurance system is organized, directly relevant statistics are not often available. However, enough ancillary information from similar risk situations may be obtained to

identify the risks and to provide preliminary estimates of the probability distributions needed to determine premiums. Because most insurance systems operate under dynamic conditions, it is important that a plan exist for collecting and analyzing insurance operating data so that the insurance system can adapt. Adaptation in this case may mean changing premiums, paying an experience-based dividend or premium refund, or modifying future policies.

In a competitive economy, market forces encourage insurers to price short-term policies so that deviations of experience from expected value behave as independent random variables. Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains. Such consistent deviations would indicate inefficiencies in the insurance market.

As a result, the classification of risks into homogeneous groups is an important function within a market-based insurance system. Experience deviations that are random indicate efficiency or equity in classification. In a competitive insurance market, the continual interaction of numerous buyers and sellers forces experimentation with classification systems as the market participants attempt to take advantage of perceived patterns of deviations. Because insurance losses may be relatively rare events, it is often difficult to identify nonrandom patterns. The cost of classification information for a refined classification system also places a bound on experimentation in this area.

For insurance systems organized to serve groups rather than individuals, the issue is no longer whether deviations in insurance experience are random for each individual. Instead, the question is whether deviations in group experience are random. Consistent deviations in experience from that expected would indicate the need for a revision in the system.

Group insurance decisions do not rest on individual expected utility comparisons. Instead, group insurance plans are based on a collective decision on whether the system increases the total welfare of the group. Group health insurance providing benefits for the employees of a firm is an example.

1.5 Optimal Insurance

The ideas outlined in Sections 1.2, 1.3, and 1.4 have been used as the foundation of an elaborate theory for guiding insurance decision makers to actions consistent with their preferences. In this section we present one of the main results from this theory and review many of the ideas introduced so far.

A decision maker has wealth of amount w and faces a loss in the next period. This loss is a random variable X. The decision maker can buy an insurance contract that will pay $I(x)$ of the loss x. To avoid an incentive to incur the loss, we assume that all ***feasible*** insurance contracts are such that $0 \leq I(x) \leq x$. We make the

simplifying assumption that all feasible insurance contracts with $E[I(X)] = \beta$ can be purchased for the same amount P.

The decision maker has formulated a utility function $u(w)$ that is consistent with his preferences for distributions of outcomes. We assume that the decision maker is risk averse, $u''(w) < 0$. We further assume that the decision maker has decided on the amount, denoted by P, to be paid for insurance. The question is: which of the insurance contracts from the class of feasible contracts with expected claims, β, and premium, P, should be purchased to maximize the expected utility of the decision maker?

One subclass of the class of feasible insurance contracts is defined as follows:

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases} \tag{1.5.1}$$

This class of contracts is characterized by the fact that claim payments do not start until the loss exceeds the ***deductible*** amount d. For losses above the deductible amount, the excess is paid under the terms of the contract. This type of contract is sometimes called ***stop-loss*** or ***excess-of-loss insurance,*** the choice depending on the application.

In the problem discussed in this section the expected claims are denoted by β. In (1.5.2) the symbol $f(x)$ denotes the p.d.f. and the symbol $F(x)$ denotes the distribution function (d.f.) associated with the random loss X:

$$\beta = \int_d^\infty (x - d) f(x) dx \tag{1.5.2A}$$

or

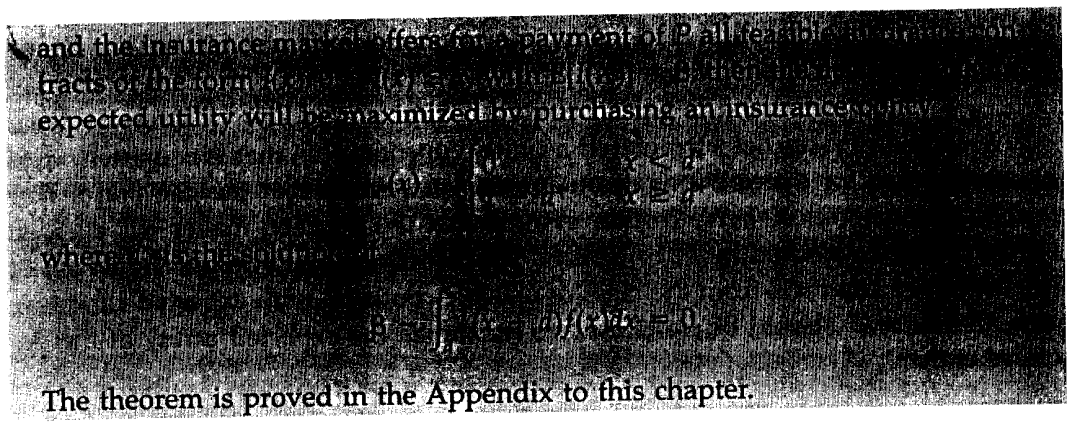
$$\beta = \int_d^\infty [1 - F(x)] dx. \tag{1.5.2B}$$

Equation (1.5.2B) is obtained from (1.5.2A) by integration by parts. When β is given, then (1.5.2) provides explicit equations for the corresponding deductible, denoted by d^*. In Exercise 1.17, it is shown that d^* exists and is unique.

The main result of this section can be stated as a theorem.

Theorem 1.5.1

If a decision maker
has wealth of amount w
is risk averse, in other words, has utility of wealth function $u(w)$ such that
$u''(w) < 0$
faces a random loss X
will spend amount P on insurance



The theorem is proved in the Appendix to this chapter.

Theorem 1.5.1 is an important result and illustrates many of the ideas developed in this chapter. However, it is instructive to consider certain limitations on its applicability. First, the ratio of premium to expected claims is the same for all available contracts. In fact, the distributions of the random variables $I(X)$ can be very different, and the provision for risk in the premium usually depends on the characteristics of the distribution of $I(X)$. Second, in Theorem 1.5.1, it is assumed that the premium P is fixed by a budget constraint. Alternatives to amount P are not considered. In Exercise 1.22, relaxation of the budget constraint is considered. Third, while the theorem indicates the form of insurance, it does not help to determine the amount P to spend. In the theorem, P is fixed.

1.6 Notes and References

Definitions and principles of actuarial science can be found in "Principles of Actuarial Science" (SOA Committee on Actuarial Principles 1992).

The role of risk in business was developed in a pioneering thesis by Willett (1951). Borch (1974) has published a series of papers applying utility theory to insurance questions. DeGroot (1970) gives a complete development of utility theory starting from basic axioms for consistency among preferences for various distributions of outcomes. DeGroot and Borch both discuss the historically important St. Petersburg paradox, outlined in Exercise 1.2. A paper by Friedman and Savage (1948) provides many insights into utility theory and human behavior.

Pratt (1964) has studied (1.3.1) and derived several theorems about premiums and utility functions. Exercise 1.10, which uses two rough approximations, is related to one of Pratt's results.

Theorem 1.5.1 on optimal insurance was proved by Arrow (1963) in the context of health insurance. The theorem in Exercise 1.21, in which the goal of insurance is to minimize the variance of retained losses, was the subject of papers by Borch (1960) and Kahn (1961). The use of the variance of losses as a measure of stability is discussed by Beard, Pentikäinen, and Pesonen (1984). Exercise 1.23 is based on their discussion.

Appendix

Lemma:
If $u''(w) < 0$ for all w in $[a, b]$, then for w and z in $[a, b]$,

$$u(w) - u(z) \leq (w - z)u'(z). \tag{1.A.1}$$

Proof:
The lemma may be established with the aid of Figure 1.3.1. Using the point slope form, a line tangent to $u(w)$ at the point $(z, u(z))$ has the equation $y - u(z) = u'(z)(w - z)$ and is above the graph of the function $u(w)$ except at the point $(z, u(z))$. Therefore,

$$u(w) - u(z) \leq u'(z)(w - z). \qquad \blacksquare$$

Figure 1.3.1 shows the case $u'(w) > 0$. The same argument holds for $u'(w) < 0$.

In Exercise 1.20 an alternative proof is required.

Proof of Theorem 1.5.1:
Let $I(x)$ be associated with an insurance policy satisfying the hypothesis of the theorem. Then from the lemma,

$$\begin{aligned} &u(w - x + I(x) - P) - u(w - x + I_{d^*}(x) - P) \\ &\leq [I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P). \end{aligned} \tag{1.A.2}$$

In addition, we claim

$$\begin{aligned} &[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \\ &\leq [I(x) - I_{d^*}(x)]u'(w - d^* - P). \end{aligned} \tag{1.A.3}$$

To establish inequality (1.A.3), we must consider three cases:

Case I. $I_{d^*}(x) = I(x)$
In this case equality holds, (1.A.3) is 0 on both sides.

Case II. $I_{d^*}(x) > I(x)$
In this case $I_{d^*}(x) > 0$ and from (1.5.1), $x - I_{d^*}(x) = d^*$. Therefore, equality holds with each side of (1.A.3) equal to $[I(x) - I_{d^*}(x)]u'(w - d^* - P)$.

Case III. $I_{d^*}(x) < I(x)$
In this case $I(x) - I_{d^*}(x) > 0$. From (1.5.1) we obtain $I_{d^*}(x) - x \geq -d^*$ and $I_{d^*}(x) - x - P \geq -d^* - P$. Therefore,

$$u'(w - x + I_{d^*}(x) - P) \leq u'(w - d^* - P)$$

since the second derivative of $u(x)$ is negative and $u'(x)$ is a decreasing function.

Therefore, in each case

$$[I(x) - I_{d^*}(x)]u'(w - x + I_{d^*}(x) - P) \leq [I(x) - I_{d^*}(x)]u'(w - P - d^*),$$

establishing inequality (1.A.3).

Now, combining inequalities (1.A.2) and (1.A.3) and taking expectations, we have

$$\begin{aligned} &\mathrm{E}[u(w - X + I(X) - P)] - \mathrm{E}[u(w - X + I_{d^*}(X) - P)] \\ &\quad \leq \mathrm{E}[I(X) - I_{d^*}(X)]u'(w - d^* - P) = (\beta - \beta)u'(w - d^* - P) = 0. \end{aligned}$$

Therefore,

$$\mathrm{E}[u(w - X + I(X) - P)] \leq \mathrm{E}[u(w - X + I_{d^*}(X) - P)]$$

and the expected utility will be maximized by selecting $I_{d^*}(x)$, the stop-loss policy. ■

Exercises

Section 1.2

1.1. Assume that a decision maker's current wealth is 10,000. Assign $u(0) = -1$ and $u(10{,}000) = 0$.

a. When facing a loss of X with probability 0.5 and remaining at current wealth with probability 0.5, the decision maker would be willing to pay up to G for complete insurance. The values for X and G in three situations are given below.

X	G
10 000	6 000
6 000	3 300
3 300	1 700

Determine three values on the decision maker's utility of wealth function u.

b. Calculate the slopes of the four line segments joining the five points determined on the graph $u(w)$. Determine the rates of change of the slopes from segment to segment.

c. Put yourself in the role of a decision maker with wealth 10,000. In addition to the given values of $u(0)$ and $u(10{,}000)$, elicit three additional values on your utility of wealth function u.

d. On the basis of the five values of your utility function, calculate the slopes and the rates of change of the slopes as done in part (b).

1.2. **St. Petersburg paradox:** Consider a game of chance that consists of tossing a coin until a head appears. The probability of a head is 0.5 and the repeated trials are independent. Let the random variable N be the number of the trial on which the first head occurs.

a. Show that the probability function (p.f.) of N is given by

$$f(n) = \left(\frac{1}{2}\right)^n \qquad n = 1, 2, 3, \ldots.$$

b. Find E[N] and Var(N).

c. If a reward of $X = 2^N$ is paid, prove that the expectation of the reward does not exist.

d. If this reward has utility $u(w) = \log w$, find E[$u(X)$].

Section 1.3

1.3. **Jensen's inequalities:**

a. Assume $u''(w) < 0$, E[X] = μ, and E[$u(X)$] exist; prove that E[$u(X)$] $\leq u(\mu)$. [Hint: Express $u(w)$ as a series around the point $w = \mu$ and terminate the expansion with an error term involving the second derivative. Note that Jensen's inequalities do not require that $u'(w) > 0$.]

b. If $u''(w) > 0$, prove that E[$u(X)$] $\geq u(\mu)$.

c. Discuss Jensen's inequalities for the special case $u(w) = w^2$. What is

$$\mathrm{E}[u(X)] - u(\mathrm{E}[X])?$$

1.4. If a utility function is such that $u'(w) > 0$ and $u''(w) > 0$, use (1.3.1) to show $G \leq \mu$. A decision maker with preferences consistent with $u''(w) > 0$ is a ***risk lover.***

1.5. Construct a geometric argument, based on a graph like that displayed in Figure 1.3.1, that if $u'(w) < 0$ and $u''(w) < 0$, then (1.3.4) follows.

1.6. Confirm that the utility function $u(w) = \log w$, $w > 0$, is the utility function of a decision maker who is risk averse for $w > 0$.

1.7. A utility function is given by

$$u(w) = \begin{cases} e^{-(w-100)^2/200} & w < 100 \\ 2 - e^{-(w-100)^2/200} & w \geq 100. \end{cases}$$

a. Is $u'(w) \geq 0$?

b. For what range of w is $u''(w) < 0$?

1.8. If one assumes, as did D. Bernoulli in his comments on the St. Petersburg paradox, that utility of wealth satisfies the differential equation

$$\frac{du(w)}{dw} = \frac{k}{w} \qquad w > 0, k > 0,$$

confirm that $u(w) = k \log w + c$.

1.9. A decision maker has utility function $u(w) = k \log w$. The decision maker has wealth w, $w > 1$, and faces a random loss X, which has a uniform distribution on the interval (0, 1). Use (1.3.1) to show that the maximum insurance premium that the decision maker will pay for complete insurance is

$$G = w - \frac{w^w}{e(w-1)^{w-1}}.$$

1.10. a. In (1.3.1) use the approximations

$$u(w - G) \cong u(w - \mu) + (\mu - G)u'(w - \mu),$$

$$u(w - x) \cong u(w - \mu) + (\mu - x)u'(w - \mu) + \frac{1}{2}(\mu - x)^2 u''(w - \mu)$$

and derive the following approximation for G:

$$G \cong \mu - \frac{1}{2}\frac{u''(w - \mu)}{u'(w - \mu)}\sigma^2.$$

b. If $u(w) = k \log w$, use the approximation developed in part (a) to obtain

$$G \cong \mu + \frac{1}{2}\frac{\sigma^2}{(w - \mu)}.$$

1.11. The decision maker has a utility function $u(w) = -e^{-\alpha w}$ and is faced with a random loss that has a chi-square distribution with n degrees of freedom. If $0 < \alpha < 1/2$, use (1.3.1) to obtain an expression for G, the maximum insurance premium the decision maker will pay, and prove that $G > n = \mu$.

1.12. Rework Example 1.3.4 for
a. $u(w) = -e^{-w/400}$
b. $u(w) = -e^{-w/150}$.

1.13. a. An insurer with net worth 100 has accepted (and collected the premium for) a risk X with the following probability distribution:

$$\Pr(X = 0) = \Pr(X = 51) = \frac{1}{2}.$$

What is the maximum amount G it should pay another insurer to accept 100% of this loss? Assume the first insurer's utility function of wealth is $u(w) = \log w$.

b. An insurer, with wealth 650 and the same utility function, $u(w) = \log w$, is considering accepting the above risk. What is the minimum amount H this insurer would accept as a premium to cover 100% of the loss?

1.14. If the complete insurance of Example 1.3.4 can be purchased for 40 and the 50% coinsurance of Example 1.3.5 can be purchased for 25, the purchase of which insurance maximizes the property owner's expected utility?

Section 1.4

1.15. A hospital expense policy is issued to a group consisting of n individuals. The policy pays B dollars each time a member of the group enters a hospital.

The group is not homogeneous with respect to the expected number of hospital admissions each year. The group may be divided into r subgroups. There are n_i individuals in subgroup i and $\Sigma_1^r n_i = n$. For subgroup i the number of annual hospital admissions for each member has a Poisson distribution with parameter λ_i, $i = 1, 2, \ldots, r$. The number of annual hospital admissions for members of the group are mutually independent.

a. Show that the expected claims payment in one year is

$$B \sum_1^r n_i \lambda_i = Bn\bar{\lambda}$$

where

$$\bar{\lambda} = \frac{\sum_1^r n_i \lambda_i}{n}.$$

b. Show that the number of hospital admissions in 1 year for the group has a Poisson distribution with parameter $n\,\bar{\lambda}$.

Section 1.5

1.16. Perform the integration by parts indicated in (1.5.2). Use the fact that if E[X] exists, if and only if, $\lim_{x\to\infty} x[1 - F(x)] = 0$.

1.17. a. Differentiate the right-hand side of (1.5.2B) with respect to d.

b. Let β be a number such that $0 < \beta < E[X]$. Show that (1.5.2) has a unique solution d^*.

1.18. Let the loss random variable X have a p.d.f. given by

$$f(x) = 0.1e^{-0.1x} \qquad x > 0.$$

a. Calculate E[X] and Var(X).

b. If $P = 5$ is to be spent for insurance to be purchased by the payment of the pure premium, show that

$$I(x) = \frac{x}{2}$$

and

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d, \text{ where } d = 10 \log 2, \end{cases}$$

both represent feasible insurance policies with pure premium $P = 5$. $I(x)$ is called ***proportional insurance.***

1.19. The loss random variable X has a p.d.f. given by

$$f(x) = \frac{1}{100} \qquad 0 < x < 100.$$

a. Calculate E[X] and Var(X).
b. Consider a proportional policy where

$$I(x) = kx \qquad 0 < k < 1,$$

and a stop-loss policy where

$$I_d(x) = \begin{cases} 0 & x < d \\ x - d & x \geq d. \end{cases}$$

Determine k and d such that the pure premium in each case is $P = 12.5$.
c. Show that $\text{Var}[X - I(X)] > \text{Var}[X - I_d(x)]$.

Appendix

1.20. Establish the lemma by using an analytic rather than a geometric argument. [Hint: Expand $u(w)$ in a series as far as a second derivative remainder around the point z and subtract $u(z)$.]

1.21. Adopt the hypotheses of Theorem 1.5.1 with respect to β and insurance contracts $I(x)$ and assume $\text{E}[X] = \mu$. Prove that

$$\text{Var}[X - I(X)] = \text{E}[(X - I(X) - \mu + \beta)^2]$$

is a minimum when $I(x) = I_{d^*}(x)$. You will be proving that for a fixed pure premium, a stop-loss insurance contract will minimize the variance of retained claims. [Hint: we may follow the proof of Theorem 1.5.1 by first proving that $x^2 - z^2 \geq (x - z)(2z)$ and then establishing that

$$[x - I(x)]^2 - [x - I_{d^*}(x)]^2 \geq [I_{d^*}(x) - I(x)][2x - 2I_{d^*}(x)]$$

$$\geq 2[I_{d^*}(x) - I(x)]d^*.$$

The final inequality may be established by breaking the proof into three cases. Alternatively, by proper choice of wealth level and utility function, the result of this exercise is a special case of Theorem 1.5.1.]

1.22. Adopt the hypotheses of Theorem 1.5.1, except remove the budget constraint; that is, assume that the decision maker will pay premium P, $0 < P \leq \text{E}[X] = \mu$, that will maximize expected utility. In addition, assume that any feasible insurance can be purchased for its expected value. Prove that the optimal insurance is $I_0(x)$. This result can be summarized by stating that full coverage is optimal in the absence of a budget constraint if insurance can be purchased for its pure premium. [Hint: Use the lemma with the role of w played by $w - x + I(x) - P$ and that of z played by $w - x + I_0(x) - \text{E}[X] = w - \mu$. Take expectations and establish that $\text{E}[u(w - X) + I(X) - P] \leq u(w - \mu)$.]

1.23. Optimality properties of stop-loss insurance were established in Theorem 1.5.1 and Exercise 1.21. These results depended on the decision criteria, the constraints, and the insurance alternatives available. In each of these developments, there was a budget constraint. Consider the situation where there is a

risk constraint and the price of insurance depends on the insurance risk as measured by the variance.

(i) The insurance premium is $E[I(X)] + f(\text{Var}[I(X)])$, where $f(w)$ is an increasing function. The amount of $f(\text{Var}[I(X)])$ can be interpreted as a ***security loading.***

(ii) The decision maker elects to retain loss $X - I(X)$ such that $\text{Var}[X - I(X)] = V \geq 0$. This requirement imposes a risk rather than a budget constraint. The constant is determined by the degree of risk aversion of the decision maker. Fixing the accepted variance, and then optimizing expected results, is a decision criterion in investment portfolio theory.

(iii) The decision maker selects $I(x)$ to minimize $f(\text{Var}[I(X)])$. The objective is to minimize the security loading, the premium paid less the expected insurance payments. Confirm the following steps:

a. $\text{Var}[I(X)] = V + \text{Var}(X) - 2\,\text{Cov}[X, X - I(X)]$.

b. The $I(x)$ that minimizes $\text{Var}[I(X)]$ and thereby $f(\text{Var}[I(X)])$ is such that the correlation coefficient between X and $X - I(x)$ is 1.

c. It is known that if two random variables W and Z have correlation coefficient 1, then $\Pr\{W = aZ + b, \text{ where } a > 0\} = 1$. In words, the probability of their joint distribution is concentrated on a line of positive slope. In part (b), the correlation coefficient of X and $X - I(X)$ was found to be 1. Thus, $X - I(X) = aX + b$, which implies that $I(X) = (1 - a)X - b$. To be a feasible insurance, $0 \leq I(x) \leq x$ or $0 \leq (1 - a)x - b \leq x$. These inequalities imply that $b = 0$ and $0 \leq 1 - a \leq 1$ and $0 \leq a \leq 1$.

d. To determine a, set the correlation coefficient of X and $X - I(X)$ equal to 1, or equivalently, their covariance equal to the product of their standard deviations. Thus, show that $a = \sqrt{V / \text{Var}(X)}$ and thus that the insurance that minimizes $f(\text{Var}[X])$ is $I(X) = [1 - \sqrt{V / \text{Var}(X)}]X$.



INDIVIDUAL RISK MODELS FOR A SHORT TERM

2.1 Introduction

In Chapter 1 we examined how a decision maker can use insurance to reduce the adverse financial impact of some types of random events. That examination was quite general. The decision maker could have been an individual seeking protection against the loss of property, savings, or income. The decision maker could have been an organization seeking protection against those same types of losses. In fact, the organization could have been an insurance company seeking protection against the loss of funds due to excess claims either by an individual or by its portfolio of insureds. Such protection is called ***reinsurance*** and is introduced in this chapter.

The theory in Chapter 1 requires a probabilistic model for the potential losses. Here we examine one of two models commonly used in insurance pricing, reserving, and reinsurance applications.

For an insuring organization, let the random loss of a segment of its risks be denoted by S. Then S is the random variable for which we seek a probability distribution. Historically, there have been two sets of postulates for distributions of S. The ***individual risk model*** defines

$$S = X_1 + X_2 + \cdots + X_n \tag{2.1.1}$$

where X_i is the loss on insured unit i and n is the number of risk units insured. Usually the X_i's are postulated to be independent random variables, because the mathematics is easier and no historical data on the dependence relationship are needed. The other model is the collective risk model described in Chapter 12.

The individual risk model in this chapter does not recognize the time value of money. This is for simplicity and is why the title refers to short terms. Chapters 4–11 cover models for long terms.

In this chapter we discuss only ***closed models;*** that is, the number of insured units n in (2.1.1) is known and fixed at the beginning of the period. If we postulate about migration in and out of the insurance system, we have an ***open model.***

2.2 Models for Individual Claim Random Variables

First, we review basic concepts with a life insurance product. In a ***one-year term life insurance*** the insurer agrees to pay an amount b if the insured dies within a year of policy issue and to pay nothing if the insured survives the year. The probability of a claim during the year is denoted by q. The claim random variable, X, has a distribution that can be described by either its probability function, p.f., or its distribution function, d.f. The p.f. is

$$f_X(x) = \Pr(X = x) = \begin{cases} 1 - q & x = 0 \\ q & x = b \\ 0 & \text{elsewhere,} \end{cases} \tag{2.2.1}$$

and the d.f. is

$$F_X(x) = \Pr(X \le x) = \begin{cases} 0 & x < 0 \\ 1 - q & 0 \le x < b \\ 1 & x \ge b. \end{cases} \tag{2.2.2}$$

From the p.f. and the definition of moments,

$$\begin{aligned} \mathrm{E}[X] &= bq, \\ \mathrm{E}[X^2] &= b^2q, \end{aligned} \tag{2.2.3}$$

and

$$\mathrm{Var}(X) = b^2q(1 - q). \tag{2.2.4}$$

These formulas can also be obtained by writing

$$X = Ib \tag{2.2.5}$$

where b is the constant amount payable in the event of death and I is the random variable that is 1 for the event of death and 0 otherwise. Thus, $\Pr(I = 0) = 1 - q$ and $\Pr(I = 1) = q$, the mean and variance of I are q and $q(1 - q)$, respectively, and the mean and variance of X are bq and $b^2q(1 - q)$ as above.

The random variable I with its $\{0, 1\}$ range is widely applicable in actuarial models. In probability textbooks it is called an ***indicator, Bernoulli random variable,*** or ***binomial random variable*** for a single trial. We refer to it as an indicator for the sake of brevity and because it indicates the occurrence, $I = 1$, or nonoccurrence, $I = 0$, of a given event.

We now seek more general models in which the amount of claim is also a random variable and several claims can occur in a period. Health, automobile, and other property and liability coverages provide immediate examples. Extending (2.2.5), we postulate that

$$X = IB \tag{2.2.6}$$

where X is the claim random variable for the period, B gives the total claim amount incurred during the period, and I is the indicator for the event that at least one claim has occurred. As the indicator for this event, I reports the occurrence ($I = 1$) or nonoccurrence ($I = 0$) of claims in this period and not the number of claims in the period. $\Pr(I = 1)$ is still denoted by q.

Let us look at several situations and determine the distributions of I and B for a model. First, consider a 1-year term life insurance paying an extra benefit in case of accidental death. To be specific, if death is accidental, the benefit amount is 50,000. For other causes of death, the benefit amount is 25,000. Assume that for the age, health, and occupation of a specific individual, the probability of an accidental death within the year is 0.0005, while the probability of a nonaccidental death is 0.0020. More succinctly,

$$\Pr(I = 1 \text{ and } B = 50{,}000) = 0.0005$$

and

$$\Pr(I = 1 \text{ and } B = 25{,}000) = 0.0020.$$

Summing over the possible values of B, we have

$$\Pr(I = 1) = 0.0025,$$

and then

$$\Pr(I = 0) = 1 - \Pr(I = 1) = 0.9975.$$

The conditional distribution of B, given $I = 1$, is

$$\Pr(B = 25{,}000 | I = 1) = \frac{\Pr(B = 25{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0020}{0.0025} = 0.8,$$

$$\Pr(B = 50{,}000 | I = 1) = \frac{\Pr(B = 50{,}000 \text{ and } I = 1)}{\Pr(I = 1)} = \frac{0.0005}{0.0025} = 0.2.$$

Let us now consider an automobile insurance providing collision coverage (this indemnifies the owner for collision damage to his car) above a 250 deductible up to a maximum claim of 2,000. For illustrative purposes, assume that for a particular individual the probability of one claim in a period is 0.15 and the chance of more than one claim is 0:

$$\Pr(I = 0) = 0.85,$$

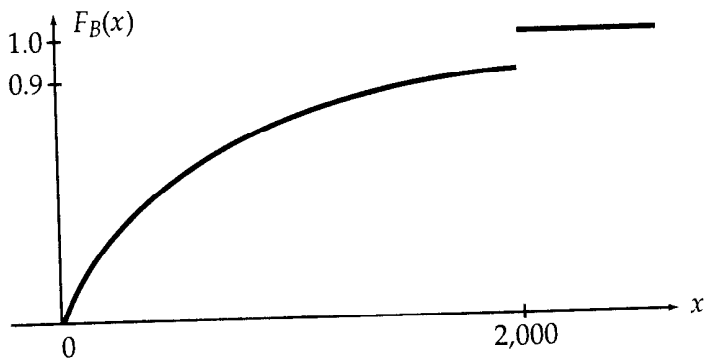
$$\Pr(I = 1) = 0.15.$$

This unrealistic assumption of no more than one claim per period is made to simplify the distribution of B. We remove that assumption in a later section after we discuss the distribution of the sum of a number of claims. Since B is the claim incurred by the insurer, rather than the amount of damage to the car, we can infer two characteristics of I and B. First, the event $I = 0$ includes those collisions in which the damage is less than the 250 deductible. The other inference is that B's distribution has a probability mass at the maximum claim size of 2,000. Assume

this probability mass is 0.1. Furthermore, assume that claim amounts between 0 and 2,000 can be modeled by a continuous distribution with a p.d.f. proportional to $1 - x/2{,}000$ for $0 < x < 2{,}000$. (In practice the continuous curve chosen to represent the distribution of claims is the result of a study of claims by size over a recent period.) Summarizing these assumptions about the conditional distribution of B, given $I = 1$, we have a mixed distribution with positive density from 0 to 2,000 and a mass at 2,000. This is illustrated in Figure 2.2.1. The d.f. of this conditional distribution is

$$\Pr(B \le x | I = 1) = \begin{cases} 0 & x \le 0 \\ 0.9\left[1 - \left(1 - \dfrac{x}{2{,}000}\right)^2\right] & 0 < x < 2{,}000 \\ 1 & x \ge 2{,}000. \end{cases}$$

Distribution Function for B, given $I = 1$



We see in Section 2.4 that the moments of the claim random variable, X, in particular the mean and variance, are extensively used. For this automobile insurance, we shall calculate the mean and the variance by two methods. First, we derive the distribution of X and use it to calculate E[X] and Var(X). Letting $F_X(x)$ be the d.f. of X, we have

$$\begin{aligned} F_X(x) &= \Pr(X \le x) = \Pr(IB \le x) \\ &= \Pr(IB \le x | I = 0)\Pr(I = 0) \\ &\quad + \Pr(IB \le x | I = 1)\Pr(I = 1). \end{aligned} \tag{2.2.7}$$

For $x < 0$,

$$F_X(x) = 0(0.85) + 0(0.15) = 0.$$

For $0 \le x < 2{,}000$,

$$F_X(x) = 1(0.85) + 0.9\left[1 - \left(1 - \frac{x}{2{,}000}\right)^2\right](0.15).$$

For $x \geq 2{,}000$,

$$F_X(x) = 1(0.85) + 1(0.15) = 1.$$

This is a mixed distribution. It has both probability masses and a continuous part as can be seen in its graph in Figure 2.2.2.

Distribution Function of $X = IB$



Corresponding to this d.f. is a combination p.f. and p.d.f. given by

$$\Pr(X = 0) = 0.85,$$
$$\Pr(X = 2{,}000) = 0.015 \tag{2.2.8}$$

with p.d.f.

$$f_X(x) = \begin{cases} F'_X(x) = 0.000135\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere.} \end{cases}$$

Moments of X can then be calculated by

$$\mathrm{E}[X^k] = 0 \times \Pr(X = 0) + (2{,}000)^k \times \Pr(X = 2{,}000) + \int_0^{2{,}000} x^k f_X(x)\, dx, \tag{2.2.9}$$

specifically,

$$\mathrm{E}[X] = 120$$

and

$$\mathrm{E}[X^2] = 150{,}000.$$

Thus,

$$\mathrm{Var}(X) = 135{,}600.$$

There are some formulas relating the moments of random variables to certain conditional expectations. General versions of these formulas for the mean and variance are

$$\mathrm{E}[W] = \mathrm{E}[\mathrm{E}[W|V]] \tag{2.2.10}$$

and

$$\text{Var}(W) = \text{Var}(\text{E}[W|V]) + \text{E}[\text{Var}(W|V)]. \tag{2.2.11}$$

In these equations we think of calculating the terms of the left-hand sides by direct use of W's distribution. In the terms on the right-hand sides, $\text{E}[W|V]$ and $\text{Var}(W|V)$ are calculated by use of W's conditional distribution for a given value of V. These components are then functions of the random variable V, and we can calculate their moments by use of V's distribution.

In many actuarial models conditional distributions are used. This makes the formulas above directly applicable. In our model, $X = IB$, we can substitute X for W and I for V to obtain

$$\text{E}[X] = \text{E}[\text{E}[X|I]] \tag{2.2.12}$$

and

$$\text{Var}(X) = \text{Var}(\text{E}[X|I]) + \text{E}[\text{Var}(X|I)]. \tag{2.2.13}$$

Now let us write

$$\mu = \text{E}[B|I = 1], \tag{2.2.14}$$

$$\sigma^2 = \text{Var}(B|I = 1), \tag{2.2.15}$$

and look at the conditional means

$$\text{E}[X|I = 0] = 0 \tag{2.2.16}$$

and

$$\text{E}[X|I = 1] = \text{E}[B|I = 1] = \mu. \tag{2.2.17}$$

Formulas (2.2.16) and (2.2.17) define $\text{E}[X|I]$ as a function of I, which can be written by the formula

$$\text{E}[X|I] = \mu I. \tag{2.2.18}$$

Hence,

$$\text{E}[\text{E}[X|I]] = \mu\text{E}[I] = \mu q \tag{2.2.19}$$

and

$$\text{Var}(\text{E}[X|I]) = \mu^2 \text{Var}(I) = \mu^2 q(1 - q). \tag{2.2.20}$$

Since $X = 0$ for $I = 0$, we have

$$\text{Var}(X|I = 0) = 0. \tag{2.2.21}$$

For $I = 1$, we have $X = B$ and

$$\text{Var}(X|I = 1) = \text{Var}(B|I = 1) = \sigma^2. \tag{2.2.22}$$

Formulas (2.2.21) and (2.2.22) can be combined as

$$\text{Var}(X|I) = \sigma^2 I. \tag{2.2.23}$$

Then

$$\mathrm{E}[\mathrm{Var}(X|I)] = \sigma^2\mathrm{E}[I] = \sigma^2 q. \tag{2.2.24}$$

Substituting (2.2.19), (2.2.20), and (2.2.24) into (2.2.12) and (2.2.13), we have

$$\mathrm{E}[X] = \mu q \tag{2.2.25}$$

and

$$\mathrm{Var}(X) = \mu^2 q(1 - q) + \sigma^2 q. \tag{2.2.26}$$

Let us now apply these formulas to calculate $\mathrm{E}[X]$ and $\mathrm{Var}(X)$ for the automobile insurance in Figure 2.2.2. Since the p.d.f. for B, given $I = 1$, is

$$f_{B|I}(x|1) = \begin{cases} 0.0009\left(1 - \dfrac{x}{2{,}000}\right) & 0 < x < 2{,}000 \\ 0 & \text{elsewhere,} \end{cases}$$

with $\Pr(B = 2{,}000|I = 1) = 0.1$, we have

$$\mu = \int_0^{2{,}000} 0.0009\, x\left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000) = 800,$$

$$\mathrm{E}[B^2|I = 1] = \int_0^{2{,}000} 0.0009\, x^2\left(1 - \frac{x}{2{,}000}\right) dx + (0.1)(2{,}000)^2 = 1{,}000{,}000,$$

and

$$\sigma^2 = 1{,}000{,}000 - (800)^2 = 360{,}000.$$

Finally, with $q = 0.15$ we obtain the following from (2.2.25) and (2.2.26):

$$\mathrm{E}[X] = 800(0.15) = 120$$

and

$$\begin{aligned} \mathrm{Var}(X) &= (800)^2(0.15)(0.85) + (360{,}000)(0.15) \\ &= 135{,}600. \end{aligned}$$

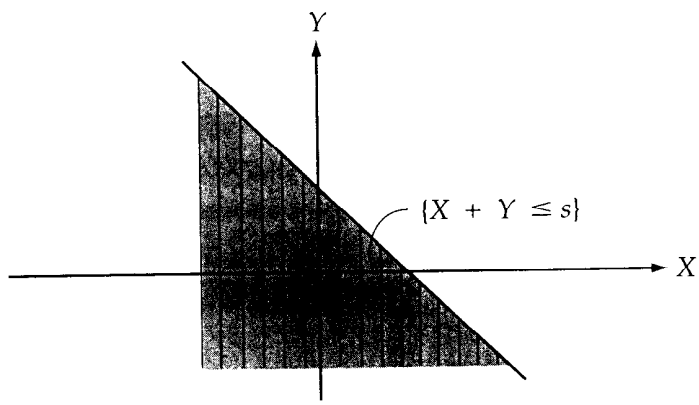
There are other possible models for B in different insurance situations. As an example, let us consider a model for the number of deaths due to crashes during an airline's year of operation. We can start with a random variable for the number of deaths, X, on a single flight and then add up a set of such random variables over the set of flights for the year. For a single flight, the event $I = 1$ will be the event of an accident during the flight. The number of deaths in the accident, B, will be modeled as the product of two random variables, L and Q, where L is the load factor, the number of persons on board at the time of the crash, and Q is the fraction of deaths among persons on board. The number of deaths B is modeled in this way since separate statistical data for the distributions of L and Q may be more readily available than are total data for B. We have $X = ILQ$. While the fraction of passengers killed in a crash and the fraction of seats occupied are probably related, L and Q might be assumed to be independent as a first approximation.

2.3 Sums of Independent Random Variables

In the individual risk model, claims of an insuring organization are modeled as the sum of the claims of many insured individuals.

The claims for the individuals are assumed to be independent in most applications. In this section we review two methods for determining the distribution of the sum of independent random variables. First, let us consider the sum of two random variables, $S = X + Y$, with the sample space shown in Figure 2.3.1.

Event $\{X + Y \leq s\}$



The line $X + Y = s$ and the region below the line represent the event

$$[S \leq X + Y \leq s].$$

Hence the d.f. of S is

$$F_S(s) = \Pr(S \leq s) = \Pr(X + Y \leq s). \tag{2.3.1}$$

For two discrete, non-negative random variables, we can use the law of total probability to write (2.3.1) as

$$\begin{aligned} F_S(s) &= \sum_{\text{all } y \leq s} \Pr(X + Y \leq s | Y = y) \Pr(Y = y) \\ &= \sum_{\text{all } y \leq s} \Pr(X \leq s - y | Y = y) \Pr(Y = y). \end{aligned} \tag{2.3.2}$$

When X and Y are independent, this last sum can be written

$$F_S(s) = \sum_{\text{all } y \leq s} F_X(s - y) f_Y(y). \tag{2.3.3}$$

The p.f. corresponding to this d.f. can be calculated by

$$f_S(s) = \sum_{\text{all } y \leq s} f_X(s - y) f_Y(y). \tag{2.3.4}$$

For continuous, non-negative random variables the formulas corresponding to (2.3.2), (2.3.3), and (2.3.4) are

$$F_S(s) = \int_0^s \Pr(X \le s - y | Y = y) f_Y(y)\, dy, \tag{2.3.5}$$

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy, \tag{2.3.6}$$

$$f_S(s) = \int_0^s f_X(s - y) f_Y(y)\, dy. \tag{2.3.7}$$

When either one, or both, of X and Y have a mixed-type distribution (typical in individual risk model applications), the formulas are analogous but more complex. For random variables that may also take on negative values, the sums and integrals in the formulas above are over all y values from $-\infty$ to $+\infty$.

In probability, the operation in (2.3.3) and (2.3.6) is called the ***convolution*** of the pair of distribution functions $F_X(x)$ and $F_Y(y)$ and is denoted by $F_X * F_Y$. Convolutions can also be defined for a pair of probability functions or probability density functions as in (2.3.4) and (2.3.7).

To determine the distribution of the sum of more than two random variables, we can use the convolution process iteratively. For $S = X_1 + X_2 + \cdots + X_n$ where the X_i's are independent random variables, F_i is the d.f. of X_i, and $F^{(k)}$ is the d.f. of $X_1 + X_2 + \cdots + X_k$, we proceed thus:

$$F^{(2)} = F_2 * F^{(1)} = F_2 * F_1$$

$$F^{(3)} = F_3 * F^{(2)}$$

$$F^{(4)} = F_4 * F^{(3)}$$

$$\vdots$$

$$F^{(n)} = F_n * F^{(n-1)}.$$

Example 2.3.1 illustrates the procedure using probability functions for three discrete random variables.

Example 2.3.1

The random variables X_1, X_2, and X_3 are independent with distributions defined by columns (1), (2), and (3) of the table below. Derive the p.f. and d.f. of

$$S = X_1 + X_2 + X_3.$$

Solution:

The notation of the previous paragraph is used in the table:

- Columns (1)–(3) are given information.
- Column (4) is derived from columns (1) and (2) by use of (2.3.4).
- Column (5) is derived from columns (3) and (4) by use of (2.3.4).

The determination of column (5) completes the determination of the distribution of S. Its d.f. in column (8) is the set of partial sums of column (5) from the top.

x	(1) $f_1(x)$	(2) $f_2(x)$	(3) $f_3(x)$	(4) $f^{(2)}(x)$	(5) $f^{(3)}(x)$	(6) $F_1(x)$	(7) $F^{(2)}(x)$	(8) $F^{(3)}(x)$
0	0.4	0.5	0.6	0.20	0.120	0.4	0.20	0.120
1	0.3	0.2	0.0	0.23	0.138	0.7	0.43	0.258
2	0.2	0.1	0.1	0.20	0.140	0.9	0.63	0.398
3	0.1	0.1	0.1	0.16	0.139	1.0	0.79	0.537
4		0.1	0.1	0.11	0.129	1.0	0.90	0.666
5			0.1	0.06	0.115	1.0	0.96	0.781
6				0.03	0.088	1.0	0.99	0.869
7				0.01	0.059	1.0	1.00	0.928
8					0.036	1.0	1.00	0.964
9					0.021	1.0	1.00	0.985
10					0.010	1.0	1.00	0.995
11					0.004	1.0	1.00	0.999
12					0.001	1.0	1.00	1.000

For illustrative purposes we include column (6), the d.f. for column (1), column (7) which can be derived directly from columns (2) and (6) by use of (2.3.3), and column (8), derived similarly from columns (3) and (7). Column (5) can then be obtained by differencing column (8). ▼

We follow with two examples involving continuous random variables.

Example 2.3.2

Let X have a uniform distribution on (0, 2) and let Y be independent of X with a uniform distribution over (0, 3). Determine the d.f. of $S = X + Y$.

Solution:

Since X and Y are continuous, we use (2.3.6):

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

and

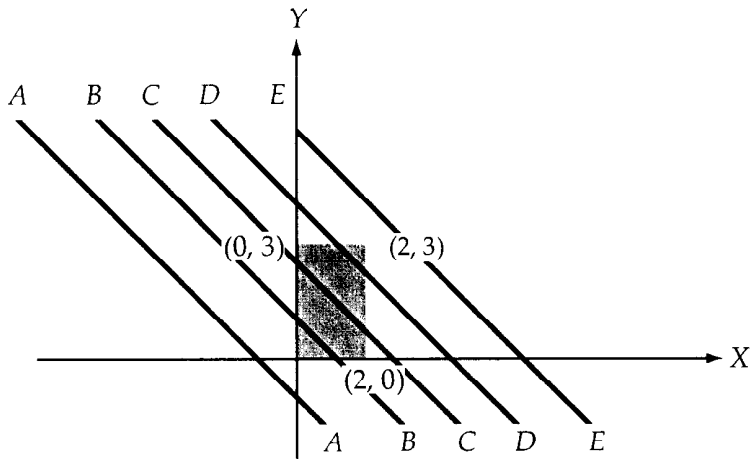
$$f_Y(y) = \begin{cases} \frac{1}{3} & 0 < y < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$F_S(s) = \int_0^s F_X(s - y) f_Y(y)\, dy.$$

The X, Y sample space is illustrated in Figure 2.3.2. The rectangular region contains all of the probability for X and Y. The event of interest, $X + Y \le s$, has been

Convolution of Two Uniform Distributions



illustrated in the figure for five values of s. For each value, the line intersects the y-axis at s and the line $x = 2$ at $s - 2$. The values of F_S for these five cases are

$$F_S(s) = \begin{cases} 0 & s < 0 \quad \text{line } A \\ \displaystyle\int_0^s \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s^2}{12} & 0 \le s < 2 \quad \text{line } B \\ \displaystyle\int_0^{s-2} 1\,\frac{1}{3}\,dy + \int_{s-2}^{s} \frac{s-y}{2}\frac{1}{3}\,dy = \frac{s-1}{3} & 2 \le s < 3 \quad \text{line } C \\ \displaystyle\int_0^{s-2} 1\,\frac{1}{3}\,dy + \int_{s-2}^{3} \frac{s-y}{2}\frac{1}{3}\,dy = 1 - \frac{(5-s)^2}{12} & 3 \le s < 5 \quad \text{line } D \\ 1 & s \ge 5 \quad \text{line } E. \end{cases}$$

▼

Example 2.3.3

Consider three independent random variables X_1, X_2, X_3. For $i = 1, 2, 3$, X_i has an exponential distribution and $E[X_i] = 1/i$. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by the convolution process.

Solution:

$$f_1(x) = e^{-x} \qquad x > 0,$$

$$f_2(x) = 2e^{-2x} \qquad x > 0,$$

$$f_3(x) = 3e^{-3x} \qquad x > 0.$$

Using (2.3.7) twice, we have

$$f^{(2)}(x) = \int_0^x f_1(x-y) f_2(y)\, dy = \int_0^x e^{-(x-y)} 2e^{-2y}\, dy$$

$$= 2e^{-x} \int_0^x e^{-y}\, dy$$

$$= 2e^{-x} - 2e^{-2x} \qquad x > 0,$$

$$f_S(x) = f^{(3)}(x) = \int_0^x f^{(2)}(x-y) f_3(y)\, dy$$

$$= \int_0^x (2e^{-(x-y)} - 2e^{-2(x-y)}) 3e^{-3y}\, dy$$

$$= 6e^{-x} \int_0^x e^{-2y}\, dy - 6e^{-2x} \int_0^x e^{-y}\, dy$$

$$= (3e^{-x} - 3e^{-3x}) - (6e^{-2x} - 6e^{-3x})$$

$$= 3e^{-x} - 6e^{-2x} + 3e^{-3x} \qquad x > 0. \quad \blacktriangledown$$

Another method to determine the distribution of the sum of random variables is based on the uniqueness of the ***moment generating function*** (m.g.f.), which, for the random variable X, is defined by $M_X(t) = \mathrm{E}[e^{tX}]$. If this expectation is finite for all t in an open interval about the origin, then $M_X(t)$ is the only m.g.f. of the distribution of X, and it is not the m.g.f. of any other distribution. This uniqueness can be used as follows. For the sum $S = X_1 + X_2 + \cdots + X_n$,

$$M_S(t) = \mathrm{E}[e^{tS}] = \mathrm{E}[e^t(X_1 + X_2 + \cdots + X_n)]$$

$$= \mathrm{E}[e^{tX_1} e^{tX_2} \cdots e^{tX_n}]. \tag{2.3.8}$$

If $X_1, X_2, \ldots, X_n$ are independent, then the expectation of the product in (2.3.8) is equal to

$$\mathrm{E}[e^{tX_1}]\ \mathrm{E}[e^{tX_2}] \cdots \mathrm{E}[e^{tX_n}]$$

so that

$$M_S(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t). \tag{2.3.9}$$

Recognition of the unique distribution corresponding to (2.3.9) would complete the determination of S's distribution. If inversion by recognition is not possible, then inversion by numerical methods may be used. (See Section 2.6.)

Example 2.3.4

Consider the random variables of Example 2.3.3. Derive the p.d.f. of $S = X_1 + X_2 + X_3$ by recognition of the m.g.f. of S.

Solution:

By (2.3.9), $M_S(t) = \left(\frac{1}{1-t}\right)\left(\frac{2}{2-t}\right)\left(\frac{3}{3-t}\right)$, which we write, by the method of partial fractions, as

$$M_S(t) = \frac{A}{1-t} + \frac{2B}{2-t} + \frac{3C}{3-t}.$$

The solution for this is $A = 3$, $B = -3$, $C = 1$. But $\beta/(\beta - t)$ is the moment generating function of an exponential distribution with parameter β, so the p.d.f. for S is

$$f_S(x) = 3(e^{-x}) - 3(2e^{-2x}) + (3e^{-3x}). \quad \blacktriangledown$$

Example 2.3.5

The ***inverse Gaussian distribution*** was developed in the study of stochastic processes. Here it is used as the distribution of B, the claim amount. It will have a similar role in risk theory in Chapters 12–14. The p.d.f. and m.g.f. associated with the inverse Gaussian distribution are given by

$$f_X(x) = \frac{\alpha}{\sqrt{2\pi\beta}} x^{-3/2} \exp\left[\frac{(\beta x - \alpha)^2}{2\beta x}\right] \qquad x > 0,$$

$$M_X(t) = \exp\left[\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

Find the distribution of $S = X_1 + X_2 + X_3 + \cdots + X_n$ where the random variables $X_1, X_2, \ldots, X_n$ are independent and have identical inverse Gaussian distributions.

Solution:

Using (2.3.9), the m.g.f. of S is given by

$$M_S(t) = [M_X(t)]^n = \exp\left[n\alpha\left(1 - \sqrt{1 - \frac{2t}{\beta}}\right)\right].$$

The m.g.f. $M_S(t)$ can be recognized and shows that S has an inverse Gaussian distribution with parameters $n\alpha$ and β. $\blacktriangledown$

2.4 Approximations for the Distribution of the Sum

The *central limit theorem* suggests a method to obtain numerical values for the distribution of the sum of independent random variables. The usual statement of the theorem is for a sequence of independent and identically distributed random variables, $X_1, X_2, \ldots$, with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. For each n, the distribution of $\sqrt{n}\,(\bar{X}_n - \mu)/\sigma$, where $\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n$, has mean 0 and variance 1. The sequence of distributions ($n = 1, 2, \ldots$) is known to approach the standard normal distribution. When n is large the theorem is applied to approximate the distribution of $\bar{X}_n$ by a normal distribution with mean μ and variance σ^2/n. Equivalently, the distribution of the sum of the n random variables is approximated by a normal distribution with mean $n\mu$ and variance $n\sigma^2$. The effectiveness of these approximations depends not only on the number of variables but also on the departure of the distribution of the summands from normality. Many elementary statistics textbooks recommend that n be at least 30 for the

approximations to be reasonable. One routine used to generate normally distributed random variables for simulation is based on the average of only 12 independent random variables uniformly distributed over (0, 1).

In many individual risk models the random variables in the sum are not identically distributed. This is illustrated by examples in the next section. The central limit theorem extends to sequences of nonidentically distributed random variables.

To illustrate some applications of the individual risk model, we use a normal approximation to the distribution of the sum of independent random variables to obtain numerical answers. If

$$S = X_1 + X_2 + \cdots + X_n,$$

then

$$\mathrm{E}[S] = \sum_{k=1}^{n} \mathrm{E}[X_k],$$

and, further, under the assumption of independence,

$$\mathrm{Var}(S) = \sum_{k=1}^{n} \mathrm{Var}(X_k).$$

For an application we need only

- Evaluate the means and variances of the individual loss random variables
- Sum them to obtain the mean and variance for the loss of the insuring organization as a whole
- Apply the normal approximation.

Illustrations of this process follow.

2.5 Applications to Insurance

In this section four examples illustrate the results of Section 2.2 and use of the normal approximation.

Example 2.5.1

A life insurance company issues 1-year term life contracts for benefit amounts of 1 and 2 units to individuals with probabilities of death of 0.02 or 0.10. The following table gives the number of individuals n_k in each of the four classes created by a benefit amount b_k and a probability of claim q_k.

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.10	1	300
4	0.10	2	500

The company wants to collect, from this population of 1,800 individuals, an amount equal to the 95th percentile of the distribution of total claims. Moreover, it wants each individual's share of this amount to be proportional to that individual's expected claim. The share for individual j with mean $E[X_j]$ would be $(1 + \theta)E[X_j]$. The 95th percentile requirement suggests that $\theta > 0$. This extra amount, $\theta E[X_j]$, is the ***security loading*** and θ is the ***relative security loading.*** Calculate θ.

Solution:

The criterion for θ is $\Pr(S \leq (1 + \theta)E[S]) = 0.95$ where $S = X_1 + X_2 + \cdots + X_{1,800}$. This probability statement is equivalent to

$$\Pr\left[\frac{S - E[S]}{\sqrt{\text{Var}(S)}} \leq \frac{\theta E[S]}{\sqrt{\text{Var}(S)}}\right] = 0.95.$$

Following the discussion of the central limit theorem in Section 2.4, we approximate the distribution of $(S - E[S])/\sqrt{\text{Var}(S)}$ by the standard normal distribution and use its 95th percentile to obtain

$$\frac{\theta E[S]}{\sqrt{\text{Var}(S)}} = 1.645.$$

It remains to calculate the mean and variance of S and to calculate θ by this equation.

For the four classes of insured individuals, we have the results given below.

			Mean	Variance	
k	q_k	b_k	$b_k q_k$	$b_k^2 q_k(1 - q_k)$	n_k
1	0.02	1	0.02	0.0196	500
2	0.02	2	0.04	0.0784	500
3	0.10	1	0.10	0.0900	300
4	0.10	2	0.20	0.3600	500

Then

$$E[S] = \sum_{j=1}^{1,800} E[X_j] = \sum_{k=1}^{4} n_k b_k q_k = 160$$

and

$$\text{Var}(S) = \sum_{j=1}^{1,800} \text{Var}(X_j) = \sum_{k=1}^{4} n_k b_k^2 q_k(1 - q_k) = 256.$$

Thus, the relative security loading is

$$\theta = 1.645 \frac{\sqrt{\text{Var}(S)}}{E[S]} = 1.645 \frac{16}{160} = 0.1645.$$

▼

Example 2.5.2

The policyholders of an automobile insurance company fall into two classes.

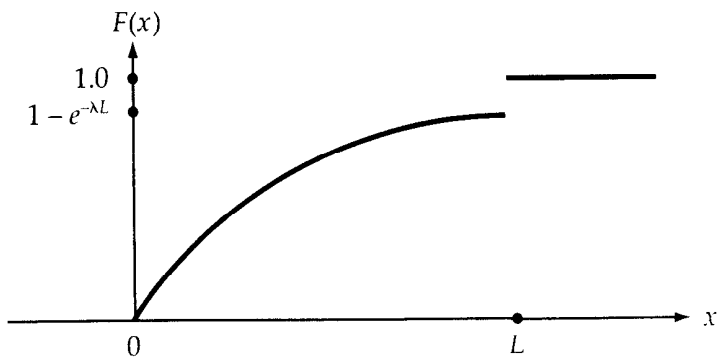
Class k	Number in Class n_k	Claim Probability q_k	Distribution of Claim Amount, B_k, Parameters of Truncated Exponential	
			λ	L
1	500	0.10	1	2.5
2	2 000	0.05	2	5.0

A truncated exponential distribution is defined by the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & 0 \le x < L \\ 1 & x \ge L. \end{cases}$$

This is a mixed distribution with p.d.f. $f(x) = \lambda e^{-\lambda x}$, $0 < x < L$, and a probability mass $e^{-\lambda L}$ at L. A graph of the d.f. appears in Figure 2.5.1.

Truncated Exponential Distribution



Again, the probability that total claims exceed the amount collected from policyholders is 0.05. We assume that the relative security loading, θ, is the same for the two classes. Calculate θ.

Solution:

This example is much like the previous one. It differs in that the claim amounts are random variables. First, we obtain formulas for the moments of the truncated exponential distribution in preparation for applying (2.2.25) and (2.2.26):

$$\mu = E[B|I = 1] = \int_0^L x\lambda e^{-\lambda x}\,dx + Le^{-\lambda L} = \frac{1 - e^{-\lambda L}}{\lambda},$$

$$E[B^2|I = 1] = \int_0^L x^2\lambda e^{-\lambda x}\,dx + L^2e^{-\lambda L} = \frac{2}{\lambda^2}(1 - e^{-\lambda L}) - \frac{2L}{\lambda}e^{-\lambda L},$$

$$\sigma^2 = E[B^2|I = 1] - (E[B|I = 1])^2 = \frac{1 - 2\lambda Le^{-\lambda L} - e^{-2\lambda L}}{\lambda^2}.$$

Using the parameter values given and applying (2.2.25) and (2.2.26), we obtain the following results.

k	q_k	μ_k	σ_k^2	Mean $q_k\mu_k$	Variance $\mu_k^2 q_k(1 - q_k) + \sigma_k^2 q_k$	n_k
1	0.10	0.9179	0.5828	0.09179	0.13411	500
2	0.05	0.5000	0.2498	0.02500	0.02436	2 000

Then S, the sum of the claims, has moments

$$E[S] = 500\ (0.09179) + 2{,}000\ (0.02500) = 95.89,$$

$$\text{Var}(S) = 500\ (0.13411) + 2{,}000\ (0.02436) = 115.78.$$

The criterion for θ is the same as in Example 2.5.1,

$$\Pr(S \leq (1 + \theta)E[S]) = 0.95.$$

Again by the normal approximation,

$$\frac{\theta E[S]}{\sqrt{\text{Var}(S)}} = 1.645$$

and

$$\theta = \frac{1.645\sqrt{115.78}}{95.89} = 0.1846.$$

▼

Example 2.5.3

A life insurance company covers 16,000 lives for 1-year term life insurance in amounts shown below.

Benefit Amount	Number Covered
10 000	8 000
20 000	3 500
30 000	2 500
50 000	1 500
100 000	500

The probability of a claim q for each of the 16,000 lives, assumed to be mutually independent, is 0.02. The company wants to set a ***retention limit.*** For each life, the retention limit is the amount below which this (the ***ceding***) company will retain

the insurance and above which it will purchase ***reinsurance*** coverage from another (the ***reinsuring***) company. For example, assume the retention limit is 20,000. The company will retain up to 20,000 on each life and purchase reinsurance for the difference between the benefit amount and 20,000 for each of the 4,500 individuals with benefit amounts in excess of 20,000. As a decision criterion, the company wants to minimize the probability that retained claims plus the amount that it pays for reinsurance will exceed 8,250,000. Reinsurance is available at a cost of 0.025 per unit of coverage (i.e., at 125% of the expected claim amount per unit, 0.02). We will consider the block of business as closed. New policies sold during the year are not to enter this decision process. Calculate the retention limit that minimizes the probability that the company's retained claims plus cost of reinsurance exceeds 8,250,000.

Partial Solution:

First, do all calculations in benefit units of 10,000. As an illustrative step, let S be the amount of retained claims paid when the retention limit is 2 (20,000). Our portfolio of retained business is given by

k	Retained Amount b_k	Number Covered n_k
1	1	8 000
2	2	8 000

and

$$E[S] = \sum_{k=1}^{2} n_k b_k q_k = 8{,}000\ (1)(0.02) + 8{,}000\ (2)(0.02) = 480$$

and

$$\text{Var}(S) = \sum_{k=1}^{2} n_k b_k^2 q_k (1 - q_k)$$

$$= 8{,}000\ (1)(0.02)(0.98) + 8{,}000\ (4)(0.02)(0.98) = 784.$$

In addition to the retained claims, S, there is the cost of reinsurance premiums. The total coverage in the plan is

$$8{,}000\ (1) + 3{,}500\ (2) + 2{,}500\ (3) + 1{,}500\ (5) + 500\ (10) = 35{,}000.$$

The retained coverage for the plan is

$$8{,}000\ (1) + 8{,}000\ (2) = 24{,}000.$$

Therefore, the total amount reinsured is $35{,}000 - 24{,}000 = 11{,}000$ and the reinsurance cost is $11{,}000(0.025) = 275$. Thus, at retention limit 2, the retained claims plus reinsurance cost is $S + 275$. The decision criterion is based on the probability that this total cost will exceed 825,

$$\Pr(S + 275 > 825) = \Pr(S > 550)$$
$$= \Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} > \frac{550 - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}}\right]$$
$$= \Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} > 2.5\right].$$

Using the normal distribution, this is approximately 0.0062. The solution is completed in Exercises 2.13 and 2.14. ▼

In Section 1.5 stop-loss insurance, which is available as a reinsurance coverage, was discussed. The expected value of the claims paid under the stop-loss reinsurance coverage can be approximated by using the normal distribution as the distribution of total claims.

Let total claims, X, have a normal distribution with mean μ and variance σ^2 and let d be the deductible of the stop-loss insurance. Then, by (1.5.2A), the expected claims equal

$$\mathrm{E}[I_d(X)] = \frac{1}{\sigma\sqrt{2\pi}} \int_d^\infty (x - d) \exp\left[\frac{-(x - \mu)^2}{2\sigma^2}\right] dx. \tag{2.5.1}$$

Changing the variable of integration to $z = (x - \mu)/\sigma$ and defining β by $d = \mu + \beta\sigma$, we obtain the following general expression for the expected value of stop-loss claims under a normal distribution assumption:

$$\mathrm{E}[I_d(X)] = \sigma\left\{\frac{\exp(-\beta^2/2)}{(2\pi)^{0.5}} - \beta[1 - \Phi(\beta)]\right\} \tag{2.5.2}$$

where $\Phi(x)$ is the distribution function for the standard normal distribution.

Example 2.5.4

Consider the portfolio of insurance contracts in Example 2.5.3. Calculate the expected value of the claims provided by a stop-loss reinsurance coverage where

a. There is no individual reinsurance and the deductible amount is 7,500,000
b. There is a retention amount of 20,000 on individual policies and the deductible amount on the business retained is 5,300,000.

Solution:

a. With no individual reinsurance and the use of 10,000 as the unit,

$$\mathrm{E}[S] = 0.02[8{,}000(1) + 3{,}500(2) + 2{,}500(3) + 1{,}500(5) + 500(10)] = 700$$

and

$$\mathrm{Var}(S) = (0.02)(0.98)[8{,}000(1) + 3{,}500(4) + 2{,}500(9) + 1{,}500(25) + 500(100)]$$
$$= 2{,}587.2$$

so

$$\sigma(S) = 50.86.$$

Then, with

$$\beta = \frac{(d - \mu)}{\sigma} = \frac{(750 - 700)}{50.86} = 0.983$$

the application of (2.5.2) gives us

$$P = 50.86[0.24608 - (0.983)(0.16280)] = 4.377.$$

This is equivalent to 43,770 in the example as posed.

b. In Example 2.5.3 we determined the mean and the variance of the aggregate claims, after imposing a 20,000 retention limit per individual, to be 480 and 784, respectively, in units of 10,000. Thus $\sigma(S) = 28$.

Then, with

$$\beta = \frac{d - \mu}{\sigma} = \frac{530 - 480}{28} = 1.786$$

the application of (2.5.2) gives us

$$P = 28[0.08100 - (1.786)(0.03707)] = 0.414.$$

This is equivalent to 4,140 in the example as posed. ▼

2.6 Notes and References

The basis of the material in Sections 2.2, 2.3, and 2.4 can be found in a number of post-calculus probability and statistics texts. Mood et al. (1974) prove the theorems given in (2.2.10) and (2.2.11). They also provide an extensive discussion of properties of the moment generating function. For a discussion of the advanced mathematical methods for deriving the distribution function that corresponds to a given moment generating function, see Bellman et al. (1966). Methods are also available to obtain the p.f. of a discrete distribution from its ***probability generating function;*** see Kornya (1983).

DeGroot (1986) provides a discussion of several conditions under which the central limit theorem holds. Kendall and Stuart (1977) give material on ***normal power expansions*** that may be viewed as modifications of the normal approximation to improve numerical results. Bowers (1967) also describes the use of normal power expansions and gives an application to approximate the distribution of present values for an annuity portfolio.

Exercises

Section 2.2

2.1. Use (2.2.3) and (2.2.4) to obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount is fixed at 10.

2.2. Obtain the mean and variance of the claim random variable X where $q = 0.05$ and the claim amount random variable B is uniformly distributed between 0 and 20.

2.3. Let X be the number of heads observed in five tosses of a true coin. Then, X true dice are thrown. Let Y be the sum of the numbers showing on the dice. Determine the mean and variance of Y. [Hint: Apply (2.2.10) and (2.2.11).]

2.4. Let X be the number showing when one true die is thrown. Let Y be the number of heads obtained when X true coins are then tossed. Calculate $E[Y]$ and $Var(Y)$.

2.5. Let X be the number obtained when one true die is tossed. Let Y be the sum of the numbers obtained when X true dice are then thrown. Calculate $E[Y]$ and $Var(Y)$.

2.6. The probability of a fire in a certain structure in a given time period is 0.02. If a fire occurs, the damage to the structure is uniformly distributed over the interval $(0, a)$ where a is its total value. Calculate the mean and variance of fire damage to the structure within the time period.

Section 2.3

2.7. Independent random variables X_k for four lives have the discrete probability functions given below.

x	$\Pr(X_1 = x)$	$\Pr(X_2 = x)$	$\Pr(X_3 = x)$	$\Pr(X_4 = x)$
0	0.6	0.7	0.6	0.9
1	0.0	0.2	0.0	0.0
2	0.3	0.1	0.0	0.0
3	0.0	0.0	0.4	0.0
4	0.1	0.0	0.0	0.1

Use a convolution process on the non-negative integer values of x to obtain $F_S(x)$ for $x = 0, 1, 2, \ldots, 13$ where $S = X_1 + X_2 + X_3 + X_4$.

2.8. Let X_i for $i = 1, 2, 3$ be independent and identically distributed with the d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

Let $S = X_1 + X_2 + X_3$.

a. Show that $F_S(x)$ is given by

$$F_S(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^3}{6} & 0 \le x < 1 \\ \dfrac{x^3 - 3(x-1)^3}{6} & 1 \le x < 2 \\ \dfrac{x^3 - 3(x-1)^3 + 3(x-2)^3}{6} & 2 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

b. Show that $E[S] = 1.5$ and $\mathrm{Var}(S) = 0.25$.

c. Evaluate the following probabilities using the d.f. of part (a):

(i) $\Pr(S \le 0.5)$

(ii) $\Pr(S \le 1.0)$

(iii) $\Pr(S \le 1.5)$.

2.9. Find the mean and variance of the inverse Gaussian distribution by using its m.g.f. as given in Example 2.3.5.

Section 2.4

2.10. Calculate the mean and variance of X and Y in Example 2.3.2. Use a normal distribution to approximate $\Pr(X + Y > 4)$. Compare this with the exact answer.

2.11. a. Use the central limit theorem to calculate b, c, and d, for given a, in the statement

$$\Pr\left(\sum_1^n X_i \ge n\mu + a\sqrt{n}\sigma\right) \cong c + b\Phi(d)$$

where the X_i's are independent and identically distributed with mean μ and variance σ^2 and $\Phi(z)$ is the d.f. of the standard normal distribution.

b. Evaluate the probabilities in Exercise 2.8(c) by use of the normal approximation developed in part (a).

2.12. A random variable U has m.g.f.

$$M_U(t) = (1 - 2t)^{-9} \qquad t < \frac{1}{2}.$$

a. Use the m.g.f. to calculate the mean and variance of U.

b. Use a normal approximation to calculate points $y_{0.05}$ and $y_{0.01}$ such that $\Pr(U > y_\epsilon) = \epsilon$.

Note the random variable U has a gamma distribution with parameters $\alpha = 9$ and $\beta = 1/2$. Gamma distributions with $\alpha = n/2$ and $\beta = 1/2$ are chi-square distributions with n degrees of freedom. Thus U has a chi-square distribution with 18 degrees of freedom. From tables of d.f.'s of chi-square distributions, we obtain $y_{0.05} = 28.869$ and $y_{0.01} = 34.805$.

Section 2.5

2.13. Calculate the probability that the total cost in Example 2.5.3 will exceed 8,250,000 if the retention limit is
a. 30,000 b. 50,000.

2.14. Calculate the retention limit that minimizes the probability of the total cost in Example 2.5.3 exceeding 8,250,000. Assume that the limit is between 30,000 and 50,000.

2.15. A fire insurance company covers 160 structures against fire damage up to an amount stated in the contract. The numbers of contracts at the different contract amounts are given below.

Contract Amount	Number of Contracts
10 000	80
20 000	35
30 000	25
50 000	15
100 000	5

Assume that for each of the structures, the probability of one claim within a year is 0.04, and the probability of more than one claim is 0. Assume that fires in the structures are mutually independent events. Furthermore, assume that the conditional distribution of the claim size, given that a claim has occurred, is uniformly distributed over the interval from 0 to the contract amount. Let N be the number of claims and let S be the amount of claims in a 1-year period.
a. Calculate the mean and variance of N.
b. Calculate the mean and variance of S.
c. What relative security loading, θ, should be used so the company can collect an amount equal to the 99th percentile of the distribution of total claims? (Use a normal approximation.)

2.16. Consider a portfolio of 32 policies. For each policy, the probability q of a claim is 1/6 and B, the benefit amount given that there is a claim, has p.d.f.

$$f(y) = \begin{cases} 2(1-y) & 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let S be the total claims for the portfolio. Using a normal approximation, estimate $\Pr(S > 4)$.



SURVIVAL DISTRIBUTIONS AND LIFE TABLES

3.1 Introduction

Chapter 1 was dedicated to showing how insurance can increase the expected utility of individuals facing random losses. In Chapter 2 simple models for single-period insurance policies were developed. The foundations of these models were Bernoulli random variables associated with the occurrence or nonoccurrence of a loss. The occurrence of a loss, in some examples, resulted in a second random process generating the amount of the loss. Chapters 4 through 8 deal primarily with models for insurance systems designed to manage random losses where the randomness is related to how long an individual will survive. In these chapters the ***time-until-death*** random variable, $T(x)$, is the basic building block. This chapter develops a set of ideas for describing and using the distribution of time-until-death and the distribution of the corresponding ***age-at-death,*** X.

We show how a distribution of the age-at-death random variable can be summarized by a ***life table.*** Such tables are useful in many fields of science. Consequently a profusion of notation and nomenclature has developed among the various professions using life tables. For example, engineers use life tables to study the reliability of complex mechanical and electronic systems. Biostatisticians use life tables to compare the effectiveness of alternative treatments of serious diseases. Demographers use life tables as tools in population projections. In this text, life tables are used to build models for insurance systems designed to assist individuals facing uncertainty about the times of their deaths. This application determines the viewpoint adopted. However, when it provides a bridge to other disciplines, notes relating the discussion to alternative applications of life tables are added.

A life table is an indispensable component of many models in actuarial science. In fact, some scholars fix the date of the beginning of actuarial science as 1693. In that year, Edmund Halley published "An Estimate of the Degrees of the Mortality of Mankind, Drawn from Various Tables of Births and Funerals at the City of

Breslau." The life table, called the Breslau Table, contained in Halley's paper remains of interest because of its surprisingly modern notation and ideas.

3.2 Probability for the Age-at-Death

In this section we formulate the uncertainty of age-at-death in probability concepts.

3.2.1 The Survival Function

Let us consider a newborn child. This newborn's age-at-death, X, is a continuous-type random variable. Let $F_X(x)$ denote the distribution function (d.f.) of X,

$$F_X(x) = \Pr(X \leq x) \qquad x \geq 0, \tag{3.2.1}$$

and set

$$s(x) = 1 - F_X(x) = \Pr(X > x) \qquad x \geq 0. \tag{3.2.2}$$

We always assume that $F_X(0) = 0$, which implies $s(0) = 1$. The function $s(x)$ is called the ***survival function*** (s.f.). For any positive x, $s(x)$ is the probability a newborn will attain age x. The distribution of X can be defined by specifying either the function $F_X(x)$ or the function $s(x)$. Within actuarial science and demography, the survival function has traditionally been used as a starting point for further developments. Within probability and statistics, the d.f. usually plays this role. However, from the properties of the d.f., we can deduce corresponding properties of the survival function.

Using the laws of probability, we can make probability statements about the age-at-death in terms of either the survival function or the distribution function. For example, the probability that a newborn dies between ages x and z $(x < z)$ is

$$\begin{aligned}\Pr(x < X \leq z) &= F_X(z) - F_X(x) \\ &= s(x) - s(z).\end{aligned}$$

3.2.2 Time-until-Death for a Person Age *x*

The conditional probability that a newborn will die between the ages x and z, given survival to age x, is

$$\begin{aligned}\Pr(x < X \leq z | X > x) &= \frac{F_X(z) - F_X(x)}{1 - F_X(x)} \\ &= \frac{s(x) - s(z)}{s(x)}.\end{aligned} \tag{3.2.3}$$

The symbol (x) is used to denote a ***life-age-x.*** The future lifetime of (x), $X - x$, is denoted by $T(x)$.

Within actuarial science, it is frequently necessary to make probability statements about $T(x)$. For this purpose, and to promote research and communication, a set of symbols, part of the International Actuarial Notation, was originally adopted by the 1898 International Actuarial Congress. Symbols for common actuarial functions and principles to guide the adoption of new symbols were established. This system has been subject to constant review and is revised or extended as necessary by the International Actuarial Association's Permanent Committee on Notation. These notational conventions are followed in this book whenever possible.

These symbols differ from those used for probability notation, and the reader may be unfamiliar with them. For example, a single-variate function that would be written $q(x)$ in probability notation is written q_x in this system. Likewise, a multivariate function is written in actuarial notation using combinations of subscripts, superscripts, and other symbols. The general rules for defining a function in actuarial notation are given in Appendix 4. The reader may want to study these forms before continuing the discussion of the future-lifetime random variable.

To make probability statements about $T(x)$, we use the notations

$$ {}_tq_x = \Pr[T(x) \le t] \qquad t \ge 0, \tag{3.2.4} $$

$$ {}_tp_x = 1 - {}_tq_x = \Pr[T(x) > t] \qquad t \ge 0. \tag{3.2.5} $$

The symbol ${}_tq_x$ can be interpreted as the probability that (x) will die within t years; that is, ${}_tq_x$ is the d.f. of $T(x)$. On the other hand, ${}_tp_x$ can be interpreted as the probability that (x) will attain age $x + t$; that is, ${}_tp_x$ is the s.f. for (x). In the special case of a life-age-0, we have $T(0) = X$ and

$$ {}_xp_0 = s(x) \qquad x \ge 0. \tag{3.2.6} $$

If $t = 1$, convention permits us to omit the prefix in the symbols defined in (3.2.4) and (3.2.5), and we have

$$ q_x = \Pr[(x) \text{ will die within 1 year}], $$

$$ p_x = \Pr[(x) \text{ will attain age } x + 1]. $$

There is a special symbol for the more general event that (x) will survive t years and die within the following u years; that is, (x) will die between ages $x + t$ and $x + t + u$. This special symbol is given by

$$ \begin{aligned} {}_{t|u}q_x &= \Pr[t < T(x) \le t + u] \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x . \end{aligned} \tag{3.2.7} $$

As before, if $u = 1$, the prefix is deleted in ${}_{t|u}q_x$, and we have ${}_{t|}q_x$.

At this point it appears there are two expressions for the probability that (x) will die between ages x and $x + u$. Formula (3.2.7) with $t = 0$ is one such expression; (3.2.3) with $z = x + u$ is a second expression. Are these two probabilities different? Formula (3.2.3) can be interpreted as the conditional probability that a newborn

will die between ages x and $z = x + u$, given survival to age x. The only information on the newborn, now at age x, is its survival to that age. Hence, the probability statement is based on a conditional distribution of survival for newborns.

On the other hand, (3.2.7) with $t = 0$ defines a probability that a life observed at age x will die between ages x and $x + u$. The observation on the life at age x might include information other than simply survival. Such information might be that the life has just passed a physical examination for insurance, or it might be that the life has commenced treatment for a serious illness. Life tables for situations where the observation of a life at age x implies more than simply survival of a newborn to age x are discussed in Section 3.8, where additional notation for those life tables is introduced. We will continue development of the theory without further reference to the distinction between (3.2.3) and (3.2.7), and we assume that until that section, observation of survival at age x will yield the same conditional distribution of survival as the hypothesis that a newborn has survived to age x; that is,

$$ {}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)}, \tag{3.2.8} $$

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)}. \tag{3.2.9} $$

Under this approach, (3.2.7), and its many special cases, can be expressed as

$$ \begin{aligned} {}_{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \left[\frac{s(x+t)}{s(x)}\right]\left[\frac{s(x+t) - s(x+t+u)}{s(x+t)}\right] \\ &= {}_tp_x \, {}_uq_{x+t}. \end{aligned} \tag{3.2.10} $$

3.2.3 Curtate-Future-Lifetime

A discrete random variable associated with the future lifetime is the number of future years completed by (x) prior to death. It is called the ***curtate-future-lifetime*** of (x) and is denoted by $K(x)$. Because $K(x)$ is the greatest integer in $T(x)$, its p.f. is

$$ \begin{aligned} \Pr[K(x) = k] &= \Pr[k \le T(x) < k+1] \\ &= \Pr[k < T(x) \le k+1] \\ &= {}_kp_x - {}_{k+1}p_x \\ &= {}_kp_x \, q_{x+k} = {}_{k|}q_x \qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{3.2.11} $$

The switching of inequalities is possible since, under our assumption that $T(x)$ is a continuous-type random variable, $\Pr[T(x) = k] = \Pr[T(x) = k+1] = 0$. Expression (3.2.11) is a special case of (3.2.7) where $u = 1$ and k is a non-negative integer. From (3.2.11) we can see that the d.f. of $K(x)$ is the step function

$$ F_{K(x)}(y) = \sum_{h=0}^{k} {}_{h|}q_x = {}_{k+1}q_x, \qquad y \ge 0 \text{ and } k \text{ is the greatest integer in } y. $$

It often follows from the context that $T(x)$ is the future lifetime of (x), in which case we may write T instead of $T(x)$. Likewise, we may write K instead of $K(x)$.

3.2.4 Force of Mortality

Formula (3.2.3) expresses, in terms of the d.f. and in terms of the survival function, the conditional probability that (0) will die between ages x and z, given survival to x. With $z - x$ held constant, say, at c, then considered as a function of x, this conditional probability describes the distribution of the probability of death in the near future (between time 0 and c) for a life of attained age x. An analogue of this function for instantaneous death can be obtained by using the density of probability of death at attained age x, that is, using (3.2.3) with $z = x + \Delta x$,

$$\Pr(x < X \leq x + \Delta x | X > x) = \frac{F_X(x + \Delta x) - F_X(x)}{1 - F_X(x)}$$

$$\cong \frac{f_X(x)\Delta x}{1 - F_X(x)}. \tag{3.2.12}$$

In this expression $F'_X(x) = f_X(x)$ is the p.d.f. of the continuous age-at-death random variable. The function

$$\frac{f_X(x)}{1 - F_X(x)}$$

in (3.2.12) has a conditional probability density interpretation. For each age x, it gives the value of the conditional p.d.f. of X at exact age x, given survival to that age, and is denoted by $\mu(x)$.

We have

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)}$$

$$= \frac{-s'(x)}{s(x)}. \tag{3.2.13}$$

The properties of $f_X(x)$ and $1 - F_X(x)$ imply that $\mu(x) \geq 0$.

In actuarial science and demography $\mu(x)$ is called the ***force of mortality.*** In reliability theory, the study of the survival probabilities of manufactured parts and systems, $\mu(x)$ is called the ***failure rate*** or ***hazard rate*** or, more fully, the ***hazard rate function.***

As is true for the s.f., the force of mortality can be used to specify the distribution of X. To obtain this result, we start with (3.2.13), change x to y, and rearrange to obtain

$$-\mu(y)\, dy = d \log s(y).$$

Integrating this expression from x to $x + n$, we have

$$-\int_x^{x+n} \mu(y)\, dy = \log\left[\frac{s(x+n)}{s(x)}\right]$$
$$= \log {}_np_x,$$

and on taking exponentials obtain

$${}_np_x = \exp[-\textstyle\int_x^{x+n} \mu(y)\, dy]. \tag{3.2.14}$$

Sometimes it is convenient to rewrite (3.2.14), with $s = y - x$, as

$${}_np_x = \exp[-\textstyle\int_0^n \mu(x+s)\, ds]. \tag{3.2.15}$$

In particular, we will change the notation to conform with that used in (3.2.6) by setting the age already lived to 0 and denoting the time of survival by x. We then have

$${}_xp_0 = s(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]. \tag{3.2.16}$$

In addition,

$$F_X(x) = 1 - s(x) = 1 - \exp[-\textstyle\int_0^x \mu(s)\, ds] \tag{3.2.17}$$

and

$$F'_X(x) = f_X(x) = \exp[-\textstyle\int_0^x \mu(s)\, ds]\ \mu(x)$$
$$= {}_xp_0\ \mu(x). \tag{3.2.18}$$

Let $F_{T(x)}(t)$ and $f_{T(x)}(t)$ denote, respectively, the d.f. and p.d.f. of $T(x)$, the future lifetime of (x). From (3.2.4) we note that $F_{T(x)}(t) = {}_tq_x$; therefore,

$$f_{T(x)}(t) = \frac{d}{dt}\, {}_tq_x$$
$$= \frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right]$$
$$= \frac{s(x+t)}{s(x)}\left[-\frac{s'(x+t)}{s(x+t)}\right]$$
$$= {}_tp_x\ \mu(x+t) \qquad t \geq 0. \tag{3.2.19}$$

Thus ${}_tp_x\ \mu(x+t)\, dt$ is the probability that (x) dies between t and $t + dt$, and

$$\int_0^\infty {}_tp_x\ \mu(x+t)\, dt = 1$$

where the upper limit on the integral is written as positive infinity (an abbreviation for integrating over all positive probability density).

It follows from (3.2.19) that

$$\frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}\, {}_tp_x = {}_tp_x\ \mu(x+t). \tag{3.2.20}$$

This equivalent form is useful in several developments in actuarial mathematics.

Since

$$\lim_{n\to\infty} {}_np_x = 0,$$

we have

$$\lim_{n\to\infty} (-\log {}_np_x) = \infty;$$

that is,

$$\lim_{n\to\infty} \int_x^{x+n} \mu(y)\, dy = \infty.$$

The developments of this section are summarized in Table 3.2.1.

The lower half of Table 3.2.1 summarizes some of the relationships among functions of general probability theory and those specific to age-at-death applications. There are many other examples where age-at-death questions can be formed in the more general probability setting. The following will illustrate this point.

Probability Theory Functions for Age-at-Death, X

	d.f. $F_X(x)$	**Survival Function** $s(x)$		**p.d.f.** $f_X(x)$	**Force of Mortality** $\mu(x)$
For	Requirements		For	Requirements	
$x < 0$	$F_X(x) = 0$	$s(x) = 1$	$x < 0$	$f_X(x) = 0$	$\mu(x) = 0$
$x = 0$	$F_X(0) = 0$	$s(0) = 1$	$x = 0$	undefined	undefined
$x \geq 0$	nondecreasing	nonincreasing	$x > 0$	$f_X(x) \geq 0$	$\mu(x) \geq 0$
$\lim_{x\to\infty}$	$F_X(\infty) = 1$	$s(\infty) = 0$	$\lim_{x\to\infty}$	$\int_0^\infty f_X(x)\, dt = 1$	$\int_0^\infty \mu(x)dx = \infty$
Functions in Terms of	Relationships				
$F_X(x)$	$F_X(x)$	$1 - F_X(x)$		$F_X'(x)$	$F_X'(x) / [1 - F_X(x)]$
$s(x)$	$1 - s(x)$	$s(x)$		$-s'(x)$	$-s'(x) / s(x)$
$f_X(x)$	$\int_0^x f_X(u)du$	$\int_x^\infty f_X(u)du$		$f_X(x)$	$f_X(x) / \int_x^\infty f_X(u)du$
$\mu(x)$	$1 - \exp[-\int_0^x \mu(t)\, dt]$	$\exp[-\int_0^x \mu(t)\, dt]$		$\mu(x)\exp[-\int_0^x \mu(t)\, dt]$	$\mu(x)$

Example 3.2.1

If $\bar{A}$ refers to the complement of the event A within the sample space and $\Pr(\bar{A}) \neq 0$, the following expresses an identity in probability theory:

$$\Pr(A \cup B) = \Pr(A) + \Pr(\bar{A})\, \Pr(B|\bar{A}).$$

Rewrite this identity in actuarial notation for the events $A = [T(x) \leq t]$ and $B = [t < T(x) \leq 1]$, $0 < t < 1$.

Solution:

$\Pr(A \cup B)$ becomes $\Pr[T(x) \le 1] = q_x$, $\Pr(A)$ is ${}_1q_x$, and $\Pr(B|\bar{A})$ is ${}_{1-1}q_{x+1}$; hence

$$q_x = {}_1q_x + {}_1p_x \, {}_{1-1}q_{x+1}.$$

▼

3.3 Life Tables

A published life table usually contains tabulations, by individual ages, of the basic functions q_x, l_x, d_x, and, possibly, additional derived functions. Before presenting such a table, we consider an interpretation of these functions that is directly related to the probability functions discussed in the preceding section.

3.3.1 Relation of Life Table Functions to the Survival Function

In (3.2.9) we expressed the conditional probability that (x) will die within t years by

$${}_tq_x = 1 - \frac{s(x+t)}{s(x)},$$

and, in particular, we have

$$q_x = 1 - \frac{s(x+1)}{s(x)}.$$

We now consider a group of l_0 newborns, $l_0 = 100{,}000$, for instance. Each newborn's age-at-death has a distribution specified by s.f. $s(x)$. In addition, we let $\mathcal{L}(x)$ denote the group's number of survivors to age x. We index these lives by $j = 1, 2, \ldots, l_0$ and observe that

$$\mathcal{L}(x) = \sum_{j=1}^{l_0} I_j$$

where I_j is an indicator for the survival of life j; that is,

$$I_j = \begin{cases} 1 & \text{if life } j \text{ survives to age } x \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{E}[I_j] = s(x)$,

$$\mathrm{E}[\mathcal{L}(x)] = \sum_{j=1}^{l_0} \mathrm{E}[I_j] = l_0 \, s(x).$$

We denote $\mathrm{E}[\mathcal{L}(x)]$ by l_x; that is, l_x represents the expected number of survivors to age x from the l_0 newborns, and we have

$$l_x = l_0 \, s(x). \tag{3.3.1}$$

Moreover, under the assumption that the indicators I_j are mutually independent, $\mathcal{L}(x)$ has a binomial distribution with parameters $n = l_0$ and $p = s(x)$. Note, however, that (3.3.1) does not require the independence assumption.

In a similar fashion, ${}_n\mathcal{Y}_x$ denotes the number of deaths between ages x and $x + n$ from among the initial l_0 lives. We denote $E[{}_n\mathcal{Y}_x]$ by ${}_nd_x$. Since a newborn has probability $s(x) - s(x + n)$ of death between ages x and $x + n$ we can, by an argument similar to that for l_x, express

$$\begin{aligned} {}_nd_x = E[{}_n\mathcal{Y}_x] &= l_0[s(x) - s(x + n)] \\ &= l_x - l_{x+n}. \end{aligned} \tag{3.3.2}$$

When $n = 1$, we omit the prefixes on ${}_n\mathcal{Y}_x$ and ${}_nd_x$.

From (3.3.1), we see that

$$-\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{1}{s(x)}\frac{ds(x)}{dx} = \mu(x) \tag{3.3.3}$$

and

$$-dl_x = l_x\,\mu(x)\,dx. \tag{3.3.4}$$

Since

$$l_x\,\mu(x) = l_0\,{}_xp_0\,\mu(x) = l_0 f_X(x),$$

the factor $l_x\,\mu(x)$ in (3.3.4) can be interpreted as the expected density of deaths in the age interval $(x, x + dx)$. We note further that

$$l_x = l_0 \exp[-\textstyle\int_0^x \mu(y)\,dy], \tag{3.3.5}$$

$$l_{x+n} = l_x \exp[-\textstyle\int_x^{x+n} \mu(y)\,dy], \tag{3.3.6}$$

$$l_x - l_{x+n} = \textstyle\int_x^{x+n} l_y\,\mu(y)\,dy. \tag{3.3.7}$$

For convenience of reference, we call this concept of l_0 newborns, each with survival function $s(x)$, a ***random survivorship group.***

3.3.2 Life Table Example

In "Life Table for the Total Population: United States, 1979–81" (Table 3.3.1), the functions ${}_tq_x$, l_x, and ${}_td_x$ are presented with $l_0 = 100{,}000$. Except for the first year of life, the value of t in the tabulated functions ${}_tq_x$ and ${}_td_x$ is 1. The other functions appearing in the table are discussed in Section 3.5.

The 1979–81 U.S. Life Table was not constructed by observing 100,000 newborns until the last survivor died. Instead, it was based on estimates of probabilities of death, given survival to various ages, derived from the experience of the entire U.S. population in the years around the 1980 census. In using the random survivorship group concept with this table, we must make the assumption that the probabilities derived from the table will be appropriate for the lifetimes of those who belong to the survivorship group.

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Age Interval	Proportion Dying	Of 100,000 Born Alive		Stationary Population*		Average Remaining Lifetime
Period of Life between Two Ages x to $x + t$	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Days						
0–1	0.00463	100 000	463	273	7 387 758	73.88
1–7	0.00246	99 537	245	1 635	7 387 485	74.22
7–28	0.00139	99 292	138	5 708	7 385 850	74.38
28–365	0.00418	99 154	414	91 357	7 380 142	74.43
Years						
0–1	0.01260	100 000	1 260	98 973	7 387 758	73.88
1–2	0.00093	98 740	92	98 694	7 288 785	73.82
2–3	0.00065	98 648	64	98 617	7 190 091	72.89
3–4	0.00050	98 584	49	98 560	7 091 474	71.93
4–5	0.00040	98 535	40	98 515	6 992 914	70.97
5–6	0.00037	98 495	36	98 477	6 894 399	70.00
6–7	0.00033	98 459	33	98 442	6 795 922	69.02
7–8	0.00030	98 426	30	98 412	6 697 480	68.05
8–9	0.00027	98 396	26	98 383	6 599 068	67.07
9–10	0.00023	98 370	23	98 358	6 500 685	66.08
10–11	0.00020	98 347	19	98 338	6 402 327	65.10
11–12	0.00019	98 328	19	98 319	6 303 989	64.11
12–13	0.00025	98 309	24	98 297	6 205 670	63.12
13–14	0.00037	98 285	37	98 266	6 107 373	62.14
14–15	0.00053	98 248	52	98 222	6 009 107	61.16
15–16	0.00069	98 196	67	98 163	5 910 885	60.19
16–17	0.00083	98 129	82	98 087	5 812 722	59.24
17–18	0.00095	98 047	94	98 000	5 714 635	58.28
18–19	0.00105	97 953	102	97 902	5 616 635	57.34
19–20	0.00112	97 851	110	97 796	5 518 733	56.40
20–21	0.00120	97 741	118	97 682	5 420 937	55.46
21–22	0.00127	97 623	124	97 561	5 323 255	54.53
22–23	0.00132	97 499	129	97 435	5 225 694	53.60
23–24	0.00134	97 370	130	97 306	5 128 259	52.67
24–25	0.00133	97 240	130	97 175	5 030 953	51.74
25–26	0.00132	97 110	128	97 046	4 933 778	50.81
26–27	0.00131	96 982	126	96 919	4 836 732	49.87
27–28	0.00130	96 856	126	96 793	4 739 813	48.94
28–29	0.00130	96 730	126	96 667	4 643 020	48.00
29–30	0.00131	96 604	127	96 541	4 546 353	47.06

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Age Interval	Proportion Dying	Of 100,000 Born Alive		Stationary Population*		Average Remaining Lifetime
Period of Life between Two Ages x to $x + t$	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
30–31	0.00133	96 477	127	96 414	4 449 812	46.12
31–32	0.00134	96 350	130	96 284	4 353 398	45.18
32–33	0.00137	96 220	132	96 155	4 257 114	44.24
33–34	0.00142	96 088	137	96 019	4 160 959	43.30
34–35	0.00150	95 951	143	95 880	4 064 940	42.36
35–36	0.00159	95 808	153	95 731	3 969 060	41.43
36–37	0.00170	95 655	163	95 574	3 873 329	40.49
37–38	0.00183	95 492	175	95 404	3 777 755	39.56
38–39	0.00197	95 317	188	95 224	3 682 351	38.63
39–40	0.00213	95 129	203	95 027	3 587 127	37.71
40–41	0.00232	94 926	220	94 817	3 492 100	36.79
41–42	0.00254	94 706	241	94 585	3 397 283	35.87
42–43	0.00279	94 465	264	94 334	3 302 698	34.96
43–44	0.00306	94 201	288	94 057	3 208 364	34.06
44–45	0.00335	93 913	314	93 756	3 114 307	33.16
45–46	0.00366	93 599	343	93 427	3 020 551	32.27
46–47	0.00401	93 256	374	93 069	2 927 124	31.39
47–48	0.00442	92 882	410	92 677	2 834 055	30.51
48–49	0.00488	92 472	451	92 246	2 741 378	29.65
49–50	0.00538	92 021	495	91 773	2 649 132	28.79
50–51	0.00589	91 526	540	91 256	2 557 359	27.94
51–52	0.00642	90 986	584	90 695	2 466 103	27.10
52–53	0.00699	90 402	631	90 086	2 375 408	26.28
53–54	0.00761	89 771	684	89 430	2 285 322	25.46
54–55	0.00830	89 087	739	88 717	2 195 892	24.65
55–56	0.00902	88 348	797	87 950	2 107 175	23.85
56–57	0.00978	87 551	856	87 122	2 019 225	23.06
57–58	0.01059	86 695	919	86 236	1 932 103	22.29
58–59	0.01151	85 776	987	85 283	1 845 867	21.52
59–60	0.01254	84 789	1 063	84 258	1 760 584	20.76
60–61	0.01368	83 726	1 145	83 153	1 676 326	20.02
61–62	0.01493	82 581	1 233	81 965	1 593 173	19.29
62–63	0.01628	81 348	1 324	80 686	1 511 208	18.58
63–64	0.01767	80 024	1 415	79 316	1 430 522	17.88
64–65	0.01911	78 609	1 502	77 859	1 351 206	17.19

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Proportion Dying					Average Remaining Lifetime
Age Interval	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval	Of 100,000 Born Alive		Stationary Population*		Average Number of Years of Life Remaining at Beginning of Age Interval
Period of Life between Two Ages x to $x + t$	$_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval $_td_x$	Years Lived in the Age Interval $_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	$\mathring{e}_x$
Years						
65–66	0.02059	77 107	1 587	76 314	1 273 347	16.51
66–67	0.02216	75 520	1 674	74 683	1 197 033	15.85
67–68	0.02389	73 846	1 764	72 964	1 122 350	15.20
68–69	0.02585	72 082	1 864	71 150	1 049 386	14.56
69–70	0.02806	70 218	1 970	69 233	978 236	13.93
70–71	0.03052	68 248	2 083	67 206	909 003	13.32
71–72	0.03315	66 165	2 193	65 069	841 797	12.72
72–73	0.03593	63 972	2 299	62 823	776 728	12.14
73–74	0.03882	61 673	2 394	60 476	713 905	11.58
74–75	0.04184	59 279	2 480	58 039	653 429	11.02
75–76	0.04507	56 799	2 560	55 520	595 390	10.48
76–77	0.04867	54 239	2 640	52 919	539 870	9.95
77–78	0.05274	51 599	2 721	50 238	486 951	9.44
78–79	0.05742	48 878	2 807	47 475	436 713	8.93
79–80	0.06277	46 071	2 891	44 626	389 238	8.45
80–81	0.06882	43 180	2 972	41 694	344 612	7.98
81–82	0.07552	40 208	3 036	38 689	302 918	7.53
82–83	0.08278	37 172	3 077	35 634	264 229	7.11
83–84	0.09041	34 095	3 083	32 553	228 595	6.70
84–85	0.09842	31 012	3 052	29 486	196 042	6.32
85–86	0.10725	27 960	2 999	26 461	166 556	5.96
86–87	0.11712	24 961	2 923	23 500	140 095	5.61
87–88	0.12717	22 038	2 803	20 636	116 595	5.29
88–89	0.13708	19 235	2 637	17 917	95 959	4.99
89–90	0.14728	16 598	2 444	15 376	78 042	4.70
90–91	0.15868	14 154	2 246	13 031	62 666	4.43
91–92	0.17169	11 908	2 045	10 886	49 635	4.17
92–93	0.18570	9 863	1 831	8 948	38 749	3.93
93–94	0.20023	8 032	1 608	7 228	29 801	3.71
94–95	0.21495	6 424	1 381	5 733	22 573	3.51
95–96	0.22976	5 043	1 159	4 463	16 840	3.34
96–97	0.24338	3 884	945	3 412	12 377	3.19
97–98	0.25637	2 939	754	2 562	8 965	3.05
98–99	0.26868	2 185	587	1 892	6 403	2.93
99–100	0.28030	1 598	448	1 374	4 511	2.82

*Stationary population is a demographic concept treated in Chapter 19.

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Proportion Dying					Average Remaining Lifetime
	Proportion of Persons Alive at Beginning of Age Interval	Of 100,000 Born Alive		Stationary Population*		Average Number of Years of Life Remaining at Beginning of Age Interval
Age Interval Period of Life between Two Ages x to $x + t$	Dying during Interval ${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	 $\mathring{e}_x$
Years						
100–101	0.29120	1 150	335	983	3 137	2.73
101–102	0.30139	815	245	692	2 154	2.64
102–103	0.31089	570	177	481	1 462	2.57
103–104	0.31970	393	126	330	981	2.50
104–105	0.32786	267	88	223	651	2.44
105–106	0.33539	179	60	150	428	2.38
106–107	0.34233	119	41	99	278	2.33
107–108	0.34870	78	27	64	179	2.29
108–109	0.35453	51	18	42	115	2.24
109–110	0.35988	33	12	27	73	2.20

*Stationary population is a demographic concept treated in Chapter 19.

Several observations about the 1979–81 U.S. Life Table are instructive.

Observations:

1. Approximately 1% of a survivorship group of newborns would be expected to die in the first year of life.
2. It would be expected that about 77% of a group of newborns would survive to age 65.
3. The maximum number of deaths within a group would be expected to occur between ages 83 and 84.
4. For human lives, there have been few observations of age-at-death beyond 110. Consequently, it is often assumed that there is an age ω such that $s(x) > 0$ for $x < \omega$, and $s(x) = 0$ for $x \geq \omega$. The age ω, if assumed, is called the ***limiting age.*** The limiting age for this table is not defined. It is clear that there is a positive probability of survival to age 110, but the table does not indicate the age ω.
5. Local minimums in the expected number of deaths occur around ages 11 and 27 and a local maximum around age 24.
6. Although the values of l_x have been rounded to integers, there is no compelling reason, according to (3.3.1), to do so.

A display such as Table 3.3.1 is the conventional method for describing the distribution of age-at-death. Alternatively, an s.f. can be described in analytic form such as $s(x) = e^{-cx}$, $c > 0$, $x \geq 0$. However, most studies of human mortality for

insurance purposes use the representation $s(x) = l_x / l_0$, as illustrated in Table 3.3.1. Since 100,000 $s(x)$ is displayed for only integer values of x, there is a need to interpolate in evaluating $s(x)$ for noninteger values. This is the subject of Section 3.6.

Example 3.3.1

On the basis of Table 3.3.1, evaluate the probability that (20) will

a. Live to 100
b. Die before 70
c. Die in the tenth decade of life.

Solution:

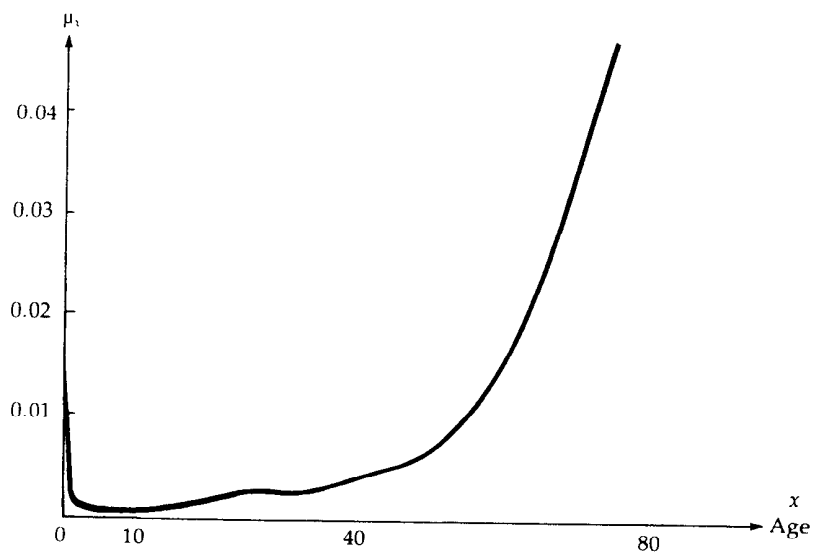
a. $$\frac{s(100)}{s(20)} = \frac{l_{100}}{l_{20}} = \frac{1{,}150}{97{,}741} = 0.0118$$

b. $$\frac{[s(20) - s(70)]}{s(20)} = 1 - \frac{l_{70}}{l_{20}} = 1 - \frac{68{,}248}{97{,}741} = 0.3017$$

c. $$\frac{[s(90) - s(100)]}{s(20)} = \frac{(l_{90} - l_{100})}{l_{20}} = \frac{(14{,}154 - 1{,}150)}{97{,}741} = 0.1330.$$ ▼

Insight into life table functions can be obtained by studying Figures 3.3.1, 3.3.2, and 3.3.3. These are drawn to be representative of current human mortality and are not taken directly from Table 3.3.1.

Graph of $\mu(x)$



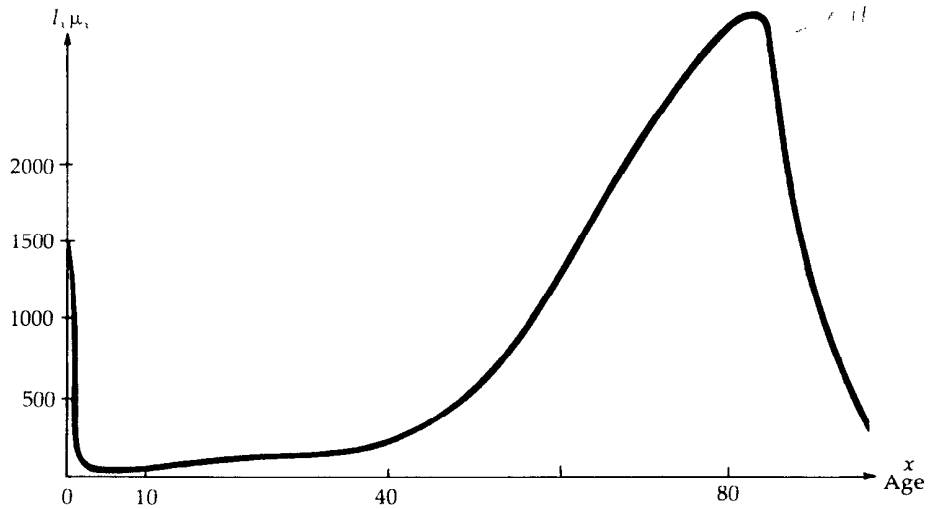
In Figure 3.3.1 note two features:

- The force of mortality is positive and the requirement

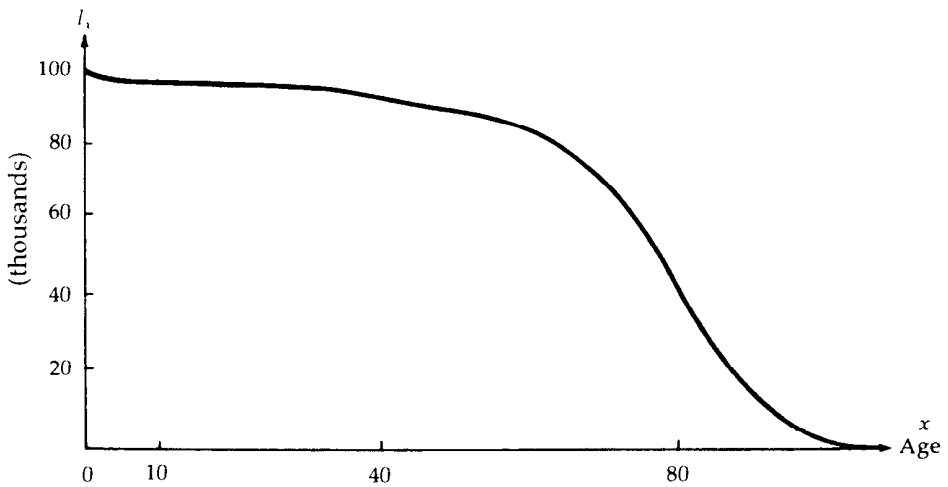
$$\int_0^\infty \mu(x)\,dx = \infty$$

  appears satisfied. (See Table 3.2.1.)
- The force of mortality starts out rather large and then drops to a minimum around age 10.

Graph of $l_x\,\mu(x)$



Graph of l_x



In Figures 3.3.2 and 3.3.3 note the following:

- The function $l_x\ \mu(x)$ is proportional to the p.d.f. of the age-at-death of a newborn. Since $l_x\ \mu(x)$ is the expected density of deaths at age x, under the random survivorship group idea, the graph of $l_x\ \mu(x)$ is called ***the curve of deaths.***
- There is a local minimum of $l_x\ \mu(x)$ at about age 10. The mode of the distribution of deaths—the age at which the maximum of the curve of deaths occurs—is around age 80.
- The function l_x is proportional to the survival function $s(x)$. It can also be interpreted as the expected number living at age x out of an initial group of size l_0.
- Local extreme points of $l_x\ \mu(x)$ correspond to points of inflection of l_x since

$$\frac{d}{dx} l_x\ \mu(x) = \frac{d}{dx}\left(-\frac{d}{dx} l_x\right) = -\frac{d^2}{dx^2} l_x .$$

3.4 The Deterministic Survivorship Group

We proceed now to a second, and nonprobabilistic, interpretation of the life table. This is rooted mathematically in the concept of decrement (negative growth) rates. As such, it is related to growth-rate applications in biology and economics. It is deterministic in nature and leads to the concept of a ***deterministic survivorship group*** or ***cohort.***

A deterministic survivorship group, as represented by a life table, has the following characteristics:

- The group initially consists of l_0 lives age 0.
- The members of the group are subject, at each age of their lives, to effective annual rates of mortality (decrement) specified by the values of q_x in the life table.
- The group is closed. No further entrants are allowed beyond the initial l_0. The only decreases come as a result of the effective annual rates of mortality (decrement).

From these characteristics it follows that the progress of the group is determined by

$$l_1 = l_0(1 - q_0) = l_0 - d_0,$$

$$l_2 = l_1(1 - q_1) = l_1 - d_1 = l_0 - (d_0 + d_1),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$l_x = l_{x-1}(1 - q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - \sum_{y=0}^{x-1} d_y$$

$$= l_0\left(1 - \frac{\sum_{y=0}^{x-1} d_y}{l_0}\right) = l_0(1 - {}_xq_0) \qquad (3.4.1)$$

where l_x is the number of lives attaining age x in the survivorship group. This chain of equalities, generated by a value l_0 called the ***radix*** and a set of q_x values, can be rewritten as

$$l_1 = l_0\, p_0,$$

$$l_2 = l_1\, p_1 = (l_0\, p_0)\, p_1,$$

$$\vdots \quad \vdots \quad \vdots$$

$$l_x = l_{x-1}\, p_{x-1} = l_0 \left(\prod_{y=0}^{x-1} p_y \right) = l_0\, {}_xp_0. \tag{3.4.2}$$

There is an analogy between the deterministic survivorship group and the model for compound interest. Table 3.4.1 is designed to summarize some of this parallelism.

Related Concepts of the Mathematics of Compound Interest and of Deterministic Survivorship Groups

Compound Interest	Survivorship Group
$A(t)$ = Size of fund at time t, time measured in years	l_x = Size of group at age x, age measured in years
Effective annual rate of interest (increment)	Effective annual rate of mortality (decrement)
$i_t = \dfrac{A(t+1) - A(t)}{A(t)}$	$q_x = \dfrac{l_x - l_{x+1}}{l_x}$
Effective n-year rate of interest, starting at time t	Effective n-year rate of mortality, starting at age x
${}_ni_t^* = \dfrac{A(t+n) - A(t)}{A(t)}$	${}_nq_x = \dfrac{l_x - l_{x+n}}{l_x}$
Force of interest at time t	Force of mortality at age x
$\delta_t = \lim\limits_{\Delta t \to 0} \left[\dfrac{A(t+\Delta t) - A(t)}{A(t)\,\Delta t} \right] = \dfrac{1}{A(t)} \dfrac{dA(t)}{dt}$	$\mu(x) = \lim\limits_{\Delta x \to 0} \left(\dfrac{l_x - l_{x+\Delta x}}{l_x\,\Delta x} \right) = -\dfrac{1}{l_x} \dfrac{dl_x}{dx}$

*There is no universally accepted symbol for an effective n-year rate of interest.

The headings of the ${}_tq_x$, l_x, and ${}_td_x$ columns in Table 3.3.1 refer to the deterministic survivorship group interpretation. Although the mathematical foundations of the random survivorship group and the deterministic survivorship group are different, the resulting functions q_x, l_x, and d_x have the same mathematical properties and subsequent analysis. The random survivorship group concept has the advantage of allowing for the full use of probability theory. The deterministic survivorship group

is conceptually simple and easy to apply but does not take account of random variation in the number of survivors.

3.5 Other Life Table Characteristics

In this section we derive expressions for some commonly used characteristics of the distributions of $T(x)$ and $K(x)$ and introduce a general method for computing several of these characteristics.

3.5.1 Characteristics

The expected value of $T(x)$, denoted by $\mathring{e}_x$, is called the ***complete-expectation-of-life.*** By definition and an integration by parts, we have

$$\begin{aligned}\mathring{e}_x = \mathrm{E}[T(x)] &= \int_0^\infty t\ {}_tp_x\ \mu(x+t)\,dt \\ &= \int_0^\infty t\ d_t(-{}_tp_x) \\ &= t(-{}_tp_x)\Big|_0^\infty + \int_0^\infty {}_tp_x\,dt. \end{aligned} \tag{3.5.1}$$

The existence of $\mathrm{E}[T(x)]$ is equivalent to the $\lim_{t\to\infty} t(-{}_tp_x) = 0$. Thus

$$\mathring{e}_x = \int_0^\infty {}_tp_x dt. \tag{3.5.2}$$

The complete-expectation-of-life at various ages is often used to compare levels of public health among different populations.

A similar integration by parts yields equivalent expressions for $\mathrm{E}[T(x)^2]$:

$$\begin{aligned}\mathrm{E}[T(x)^2] &= \int_0^\infty t^2\ {}_tp_x\ \mu(x+t)\,dt \\ &= 2\int_0^\infty t\ {}_tp_x\,dt. \end{aligned} \tag{3.5.3}$$

This result is useful in the calculation of $\mathrm{Var}\,[T(x)]$ by

$$\begin{aligned}\mathrm{Var}[T(x)] &= \mathrm{E}[T(x)^2] - \mathrm{E}[T(x)]^2 \\ &= 2\int_0^\infty t\ {}_tp_x\,dt - \mathring{e}_x^2. \end{aligned} \tag{3.5.4}$$

In these developments, we assume that $\mathrm{E}[T(x)]$ and $\mathrm{E}[T(x)^2]$ exist. One can construct s.f.'s such as $s(x) = (1+x)^{-1}$ where this would not be true.

Other characteristics of the distribution of $T(x)$ can be determined. The ***median future lifetime*** of (x), to be denoted by $m(x)$, can be found by solving

$$\Pr[T(x) > m(x)] = \frac{1}{2}$$

or

$$\frac{s[x + m(x)]}{s(x)} = \frac{1}{2} \tag{3.5.5}$$

for $m(x)$. In particular, $m(0)$ is given by solving $s[m(0)] = 1/2$. We can also find the ***mode*** of the distribution of $T(x)$ by locating the value of t that yields a maximum value of ${}_tp_x\,\mu(x + t)$.

The expected value of $K(x)$ is denoted by e_x and is called the ***curtate-expectation-of-life.*** By definition and use of summation by parts as described in Appendix 5, we have

$$\begin{aligned} e_x = \mathrm{E}[K] &= \sum_{k=0}^{\infty} k\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k\ \Delta(-{}_kp_x) \\ &= k(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} {}_{k+1}p_x. \end{aligned} \tag{3.5.6}$$

Again, the existence of $\mathrm{E}[K(x)]$ is equivalent to the $\lim_{k\to\infty} k(-{}_kp_x) = 0$. Thus, with a change of the summation variable,

$$e_x = \sum_{k=1}^{\infty} {}_kp_x. \tag{3.5.7}$$

Following the outline used for the continuous model and using summation by parts, we have

$$\begin{aligned} \mathrm{E}[K(x)^2] &= \sum_{k=0}^{\infty} k^2\ {}_kp_x\ q_{x+k} \\ &= \sum_{k=0}^{\infty} k^2\ \Delta(-{}_kp_x) \\ &= k^2(-{}_kp_x)\Big|_0^{\infty} + \sum_{k=0}^{\infty} (\Delta k^2)({}_{k+1}p_x). \end{aligned} \tag{3.5.8}$$

The existence of $\mathrm{E}[K(x)^2]$ is equivalent to $\lim_{k\to\infty} k^2(-{}_kp_x) = 0$. With a change of the summation variable,

$$\mathrm{E}[K(x)^2] = \sum_{k=0}^{\infty} (2k + 1)\ {}_{k+1}p_x = \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x \tag{3.5.9}$$

Now,

$$\begin{aligned} \mathrm{Var}(K) &= \mathrm{E}[K^2] - \mathrm{E}[K]^2 \\ &= \sum_{k=1}^{\infty} (2k - 1)\ {}_kp_x - e_x^2. \end{aligned} \tag{3.5.10}$$

To complete the discussion of some of the entries in Table 3.3.1, we must define additional functions. The symbol L_x denotes the total expected number of years lived between ages x and $x + 1$ by survivors of the initial group of l_0 lives. We have

$$L_x = \int_0^1 t\, l_{x+t}\, \mu(x + t)\, dt + l_{x+1} \tag{3.5.11}$$

where the integral counts the years lived of those who die between ages x and $x + 1$, and the term l_{x+1} counts the years lived between ages x and $x + 1$ by those who survive to age $x + 1$. Integration by parts yields

$$\begin{aligned} L_x &= -\int_0^1 t\, dl_{x+t} + l_{x+1} \\ &= -t\, l_{x+t}\Big|_0^1 + \int_0^1 l_{x+t}\, dt + l_{x+1} \\ &= \int_0^1 l_{x+t}\, dt. \end{aligned} \tag{3.5.12}$$

The function L_x is also used in defining the ***central-death-rate*** over the interval from x to $x + 1$, denoted by m_x where

$$m_x = \frac{\int_0^1 l_{x+t}\, \mu(x + t)\, dt}{\int_0^1 l_{x+t}\, dt} = \frac{l_x - l_{x+1}}{L_x}. \tag{3.5.13}$$

An application of this function is found in Chapter 10.

The definitions for m_x and L_x can be extended to age intervals of length other than one:

$$\begin{aligned} {}_nL_x &= \int_0^n t\, l_{x+t}\, \mu(x + t)\, dt + n l_{x+n} \\ &= \int_0^n l_{x+t}\, dt, \end{aligned} \tag{3.5.14}$$

$${}_nm_x = \frac{\int_0^n l_{x+t}\, \mu(x + t)\, dt}{\int_0^n l_{x+t}\, dt} = \frac{l_x - l_{x+n}}{{}_nL_x}. \tag{3.5.15}$$

For the random survivorship group, ${}_nL_x$ is the total expected number of years lived between ages x and $x + n$ by the survivors of the initial group of l_0 lives and ${}_nm_x$ is the average death rate experienced by this group over the interval $(x, x + n)$.

The symbol T_x denotes the total number of years lived beyond age x by the survivorship group with l_0 initial members. We have

$$T_x = \int_0^\infty t\, l_{x+t}\, \mu(x+t)\, dt$$

$$= -\int_0^\infty t\, d_t\, l_{x+t}$$

$$= \int_0^\infty l_{x+t}\, dt. \tag{3.5.16}$$

The final expression can be interpreted as the integral of the total time lived between ages $x + t$ and $x + t + dt$ by the l_{x+t} lives who survive to that age interval. We also recognize T_x as the limit of ${}_nL_x$ as n goes to infinity.

The average number of years of future lifetime of the l_x survivors of the group at age x is given by

$$\frac{T_x}{l_x} = \frac{\int_0^\infty l_{x+t}\, dt}{l_x}$$

$$= \int_0^\infty {}_tp_x\, dt$$

$$= \mathring{e}_x,$$

as determined in (3.5.1) and (3.5.2).

We can express the average number of years lived between x and $x + n$ by the l_x survivors at age x as

$$\frac{{}_nL_x}{l_x} = \frac{\int_0^n l_{x+t}\, dt}{l_x}$$

$$= \frac{T_x - T_{x+n}}{l_x}$$

$$= \int_0^n {}_tp_x\, dt. \tag{3.5.17}$$

This function is the ***n-year temporary complete life expectancy*** of (x) and is denoted by $\mathring{e}_{x:\overline{n}|}$. (See Exercise 3.16.)

A final function, related to the interpretation of the life table developed in this section, is the average number of years lived between ages x and $x + 1$ by those of the survivorship group who die between those ages. This function is denoted by $a(x)$ and is defined by

$$a(x) = \frac{\int_0^1 t\, l_{x+t}\, \mu(x+t)\, dt}{\int_0^1 l_{x+t}\, \mu(x+t)\, dt}. \tag{3.5.18}$$

For the probabilistic view of the life table, we would have

$$a(x) = \frac{\int_0^1 t\, {}_tp_x\, \mu(x+t)\, dt}{\int_0^1 {}_tp_x\, \mu(x+t)\, dt} = \mathrm{E}[T|T < 1].$$

If we assume that

$$l_{x+t}\, \mu(x+t)\, dt = d_x\, dt \qquad 0 \le t \le 1,$$

that is, if deaths are uniformly distributed in the year of age, we have

$$a(x) = \int_0^1 t\, dt = \frac{1}{2}.$$

This is the usual approximation for $a(x)$, except for young and old years of age where Figure 3.3.2 shows that the assumption may be inappropriate.

Example 3.5.1

Show that

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}$$

and

$$L_x \cong \frac{l_x + l_{x+1}}{2}.$$

Solution:

From (3.5.11), (3.5.12), and (3.5.18), we have

$$a(x) = \frac{L_x - l_{x+1}}{l_x - l_{x+1}}$$

or

$$L_x = a(x)\, l_x + [1 - a(x)]\, l_{x+1}.$$

The formula

$$L_x \cong \frac{l_x + l_{x+1}}{2}$$

can be justified by using the trapezoidal rule for approximate integration on (3.5.12). ▼

Key life table terminology, defined in Sections 3.3–3.5, is summarized as part of Table 3.9.1 in Section 3.9.

3.5.2 Recursion Formulas

Example 3.5.1 illustrates the use of a numerical analysis technique to evaluate a life table characteristic. The trapezoidal rule for approximate integration is used. The calculation of complete and curtate expectations-of-life can be used to illustrate another computational tool called ***recursion formulas.*** The application of recursion formulas in this book typically involves one of two forms:

Backward Recursion Formula

$$u(x) = c(x) + d(x)\, u(x+1) \tag{3.5.19}$$

or

Forward Recursion Formula

$$u(x+1) = -\frac{c(x)}{d(x)} + \frac{1}{d(x)}\, u(x). \tag{3.5.20}$$

The variable x is usually a non-negative integer.

To evaluate a function $u(x)$, for a domain of non-negative integer values of x, we need to have available values of $c(x)$ and $d(x)$ and a starting value of $u(x)$. This procedure is used in subsequent chapters and is illustrated in Table 3.5.1 where backward recursion formulas are developed to compute e_x and $\mathring{e}_x$.

Backward Recursion Formulas for e_x and $\mathring{e}_x$

Step	e_x	$\mathring{e}_x$
1. Basic equation	$e_x = \sum_{k=1}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^{\infty} {}_sp_x\, ds$
2. Separate the operation	$e_x = p_x + \sum_{k=2}^{\infty} {}_kp_x$	$\mathring{e}_x = \int_0^1 {}_sp_x\, ds + \int_1^{\infty} {}_sp_x\, ds$
3. Factor p_x and change variable in the operation	$e_x = p_x + p_x \sum_{k=1}^{\infty} {}_kp_{x+1}$ $= p_x + p_x\, e_{x+1}$	$\mathring{e}_x = \int_0^1 {}_sp_x\, ds + p_x \int_0^{\infty} {}_tp_{x+1}\, dt$ $= \int_0^1 {}_sp_x\, ds + p_x\, \mathring{e}_{x+1}$
4. Recursion formula[a]	$u(x) = e_x$, $c(x) = p_x$ $d(x) = p_x$	$u(x) = \mathring{e}_x$, $c(x) = \int_0^1 {}_sp_x\, ds$ $d(x) = p_x$
5. Starting value[b]	$e_\omega = u(\omega) = 0$	$\mathring{e}_\omega = u(x) = 0$

[a] The integral $c(x) = \int_0^1 {}_sp_x\, ds$ can be evaluated using the trapezoidal rule as $c(x) = (1 + p_x)/2$.
[b] From Section 3.3.1 we have $s(x) = 0$, $x \geq \omega$, and $s(x) > 0$, $x < \omega$. In this development we will assume that ω is an integer.

3.6 Assumptions for Fractional Ages

In this chapter we have discussed the continuous random variable remaining lifetime, T, and the discrete random variable curtate-future-lifetime, K. The life table developed in Section 3.3 specifies the probability distribution of K completely. To specify the distribution of T, we must postulate an analytic form or adopt a life table and an assumption about the distribution between integers.

We will examine three assumptions that are widely used in actuarial science. These will be stated in terms of the s.f. and in a form to show the nature of interpolation over the interval $(x, x + 1)$ implied by each assumption. In each statement, x is an integer and $0 \le t \le 1$. The assumptions are the following:

- ***Linear interpolation:*** $s(x + t) = (1 - t)\, s(x) + t\, s(x + 1)$. This is known as the ***uniform distribution*** or, perhaps more properly, a uniform distribution of deaths assumption within each year of age. Under this assumption ${}_tp_x$ is a linear function.
- ***Exponential interpolation,*** or linear interpolation on $\log s(x + t)$: $\log s(x + t) = (1 - t) \log s(x) + t \log s(x + 1)$. This is consistent with the assumption of a ***constant force*** of mortality within each year of age. Under this assumption ${}_tp_x$ is exponential.
- ***Harmonic interpolation:*** $1 / s(x + t) = (1 - t) / s(x) + t / s(x + 1)$. This is what is known as the ***hyperbolic*** (historically ***Balducci****) assumption, for under it ${}_tp_x$ is a hyperbolic curve.

With these basic definitions, formulas can be derived for other standard probability functions in terms of life table probabilities. These results are presented in Table 3.6.1. Note that we just as well could have elected to propose equivalent definitions in terms of the p.d.f., the d.f., or the force of mortality.

The derivations of the entries in Table 3.6.1 are exercises in substituting the stated assumption about $s(x + t)$ into the appropriate formulas of Sections 3.2 and 3.3. We will illustrate the process for the uniform distribution of deaths, an assumption that is used extensively throughout this text.

To derive the first entry in the uniform distribution column, we start with

$$ {}_tq_x = \frac{s(x) - s(x + t)}{s(x)} \qquad 0 \le t \le 1, $$

then substitute for $s(x + t)$,

$$ {}_tq_x = \frac{s(x) - [(1 - t)\, s(x) + t\, s(x + 1)]}{s(x)} = \frac{t\, [s(x) - s(x + 1)]}{s(x)} = t q_x . $$

For the second entry, we use (3.2.13) and

$$ \mu(x + t) = -\frac{s'(x + t)}{s(x + t)}; $$

*This assumption is named after G. Balducci, an Italian actuary, who pointed out its role in the traditional actuarial method of constructing life tables.

Probability Theory Functions for Fractional Ages

Function	Assumption		
	Uniform Distribution	Constant Force	Hyperbolic
${}_tq_x$	tq_x	$1 - p_x^t$	$\frac{tq_x}{1 - (1 - t)q_x}$
$\mu(x + t)$	$\frac{q_x}{1 - tq_x}$	$-\log p_x$	$\frac{q_x}{1 - (1 - t)q_x}$
${}_{1-t}q_{x+t}$	$\frac{(1 - t)\,q_x}{1 - tq_x}$	$1 - p_x^{1-t}$	$(1 - t)q_x$
${}_yq_{x+t}$	$\frac{yq_x}{1 - tq_x}$	$1 - p_x^y$	$\frac{yq_x}{1 - (1 - y - t)q_x}$
${}_tp_x$	$1 - tq_x$	p_x^t	$\frac{p_x}{1 - (1 - t)q_x}$
${}_tp_x\mu(x + t)$	q_x	$-p_x^t \log p_x$	$\frac{q_x\,p_x}{[1 - (1 - t)q_x]^2}$

Note that, in this table, x is an integer, $0 < t < 1$, $0 \le y \le 1$, $y + t \le 1$. For rows one, three, four, and five, the relationships also hold for $t = 0$ and $t = 1$.

then, substituting for $s(x + t)$, we have

$$\mu(x + t) = \frac{[s(x) - s(x + 1)]}{[(1 - t)\,s(x) + t\,s(x + 1)]}.$$

Dividing both numerator and denominator of the right-hand side by $s(x)$ yields

$$\mu(x + t) = \frac{q_x}{(1 - tq_x)}.$$

The third entry is the special case of the fourth entry with $y = 1 - t$.

For the fourth entry we start with

$${}_yq_{x+t} = \frac{s(x + t) - s(x + t + y)}{s(x + t)},$$

then substitute for $s(x + t)$ and $s(x + t + y)$ to obtain

$$\begin{aligned} {}_yq_{x+t} &= \frac{[(1 - t)\,s(x) + t\,s(x + 1)] - [(1 - t - y)\,s(x) + (t + y)\,s(x + 1)]}{(1 - t)\,s(x) + t\,s(x + 1)} \\ &= \frac{y[s(x) - s(x + 1)]\,/\,s(x)}{\{s(x) - t[s(x) - s(x + 1)]\}\,/\,s(x)} \\ &= \frac{yq_x}{1 - tq_x}. \end{aligned}$$

The fifth entry is the complement of the first, and the final entry for the uniform distribution column is the product of the second and fifth entries.

If, as before, x is an integer, insight can be obtained by defining a random variable $S = S(x)$ by

$$T = K + S \tag{3.6.1}$$

where T is time-until-death, K is the curtate-future-lifetime, and S is the random variable representing the fractional part of a year lived in the year of death. Since K is a non-negative integer random variable and S is a continuous-type random variable with all of its probability mass on the interval (0, 1), we can examine their joint distribution by writing

$$\begin{aligned}\Pr[(K = k) \cap (S \le s)] &= \Pr(k < T \le k + s) \\ &= {}_kp_x \, {}_sq_{x+k}.\end{aligned}$$

Now, using the expression for ${}_sq_{x+k}$ under the uniform distribution assumption as shown in Table 3.6.1, we have

$$\begin{aligned}\Pr[(K = k) \cap (S \le s)] &= {}_kp_x \, s \, q_{x+k} \\ &= {}_kq_x \, s \\ &= \Pr(K = k) \Pr(S \le s).\end{aligned} \tag{3.6.2}$$

Therefore, the joint probability involving K and S can be factored into separate probabilities of K and S. It follows that, under the uniform distribution of deaths assumption, the random variables K and S are independent. Since $\Pr(S \le s) = s$ is the d.f. of a uniform distribution on (0, 1), S has such a uniform distribution.

Example 3.6.1

Under the constant force of mortality assumption, are the random variables K and S independent?

Solution:

Using entries from Table 3.6.1 for the constant force assumption, we obtain

$$\begin{aligned}\Pr[(K = k) \cap (S \le s)] &= {}_kp_x \, {}_sq_{x+k} \\ &= {}_kp_x \, [1 - (p_{x+k})^s].\end{aligned}$$

To discuss this result, we distinguish two cases:

- If p_{x+k} is not independent of k, we cannot factor the joint probability of K and S into separate probabilities. We conclude that K and S are not independent.
- In the special case where $p_{x+k} = p_x$, a constant,

$$\begin{aligned}\Pr[(K = k) \cap (S \le s)] &= p_x^k(1 - p_x^s) = \frac{(1 - p_x)p_x^k(1 - p_x^s)}{(1 - p_x)} \\ &= \Pr(K = k) \Pr(S \le s).\end{aligned}$$

For this special case we conclude that K and S are independent under the constant force assumption. ▼

Example 3.6.2

Under the assumption of uniform distribution of deaths, show that

a. $\mathring{e}_x = e_x + \frac{1}{2}$

b. $\text{Var}(T) = \text{Var}(K) + \frac{1}{12}$.

Solution:

a. $\mathring{e}_x = \text{E}[T] = \text{E}[K + S]$
$= \text{E}[K] + \text{E}[S]$
$= e_x + \frac{1}{2}$.

b. $\text{Var}(T) = \text{Var}(K + S)$.
From the independence of K and S, under the uniform distribution assumption, it follows that

$$\text{Var}(T) = \text{Var}(K) + \text{Var}(S).$$

Further, since S is uniformly distributed over (0, 1),

$$\text{Var}(T) = \text{Var}(K) + \frac{1}{12}.$$ ▼

3.7 Some Analytical Laws of Mortality

There are three principal justifications for postulating an analytic form for mortality or survival functions. The first is philosophical. Many phenomena studied in physics can be explained efficiently by simple formulas. Therefore, using biological arguments, some authors have suggested that human survival is governed by an equally simple law. The second justification is practical. It is easier to communicate a function with a few parameters than it is to communicate a life table with perhaps 100 parameters or mortality probabilities. In addition, some of the analytic forms have elegant properties that are convenient in evaluating probability statements that involve more than one life. The third justification for a simple analytic survival function is the ease of estimating a few parameters of the function from mortality data.

The support for simple analytic survival functions has declined in recent years. Many feel that the belief in universal laws of mortality is naive. With the increasing speed and storage capacity of computers, the advantages of some analytic forms in computations involving more than one life are no longer of great importance. Nevertheless, some interesting research has recently reiterated the biological arguments for analytic laws of mortality.

In Table 3.7.1, several families of simple analytic mortality and survival functions, corresponding to various postulated laws, are displayed. The names of the originators of the laws and the dates of publication are included for identification purposes.

Mortality and Survival Functions under Various Laws

Originator	$\mu(x)$	$s(x)$	Restrictions
De Moivre (1729)	$(\omega - x)^{-1}$	$1 - \frac{x}{\omega}$	$0 \le x < \omega$
Gompertz (1825)	Bc^x	$\exp[-m(c^x - 1)]$	$B > 0, c > 1, x \ge 0$
Makeham (1860)	$A + Bc^x$	$\exp[-Ax - m(c^x - 1)]$	$B > 0, A \ge -B, c > 1, x \ge 0$
Weibull (1939)	kx^n	$\exp(-ux^{n+1})$	$k > 0, n > 0, x \ge 0$

Note:

- The special symbols are defined as

$$m = \frac{B}{\log c}, \qquad u = \frac{k}{(n+1)}.$$

- Gompertz's law is a special case of Makeham's law with $A = 0$.
- If $c = 1$ in Gompertz's and Makeham's laws, the exponential (constant force) distribution results.
- In connection with Makeham's law, the constant A has been interpreted as capturing the accident hazard, and the term Bc^x as capturing the hazard of aging.

The entries in the $s(x)$ column of Table 3.7.1 were obtained by substituting into (3.2.16). For example, for Makeham's law, we have

$$\begin{aligned} s(x) &= \exp[-\textstyle\int_0^x (A + Bc^s)ds] \\ &= \exp\left[-Ax - B\,\frac{(c^x - 1)}{\log c}\right] \\ &= \exp[-Ax - m\,(c^x - 1)] \end{aligned}$$

where $m = B / \log c$.

Two objectives governed the development of a mortality table for computational purposes in the examples and exercises of this book. One objective was to have mortality rates in the middle of the range of variation for groups, such variation caused by factors such as residence, gender, insured status, annuity status, marital status, and occupation. The second objective was to have a Makeham law at most ages to illustrate how calculations for multiple lives can be performed.

The Illustrative Life Table in Appendix 2A is based on the Makeham law for ages 13 and greater,

$$1{,}000\ \mu(x) = 0.7 + 0.05\ (10^{0.04})^x. \tag{3.7.1}$$

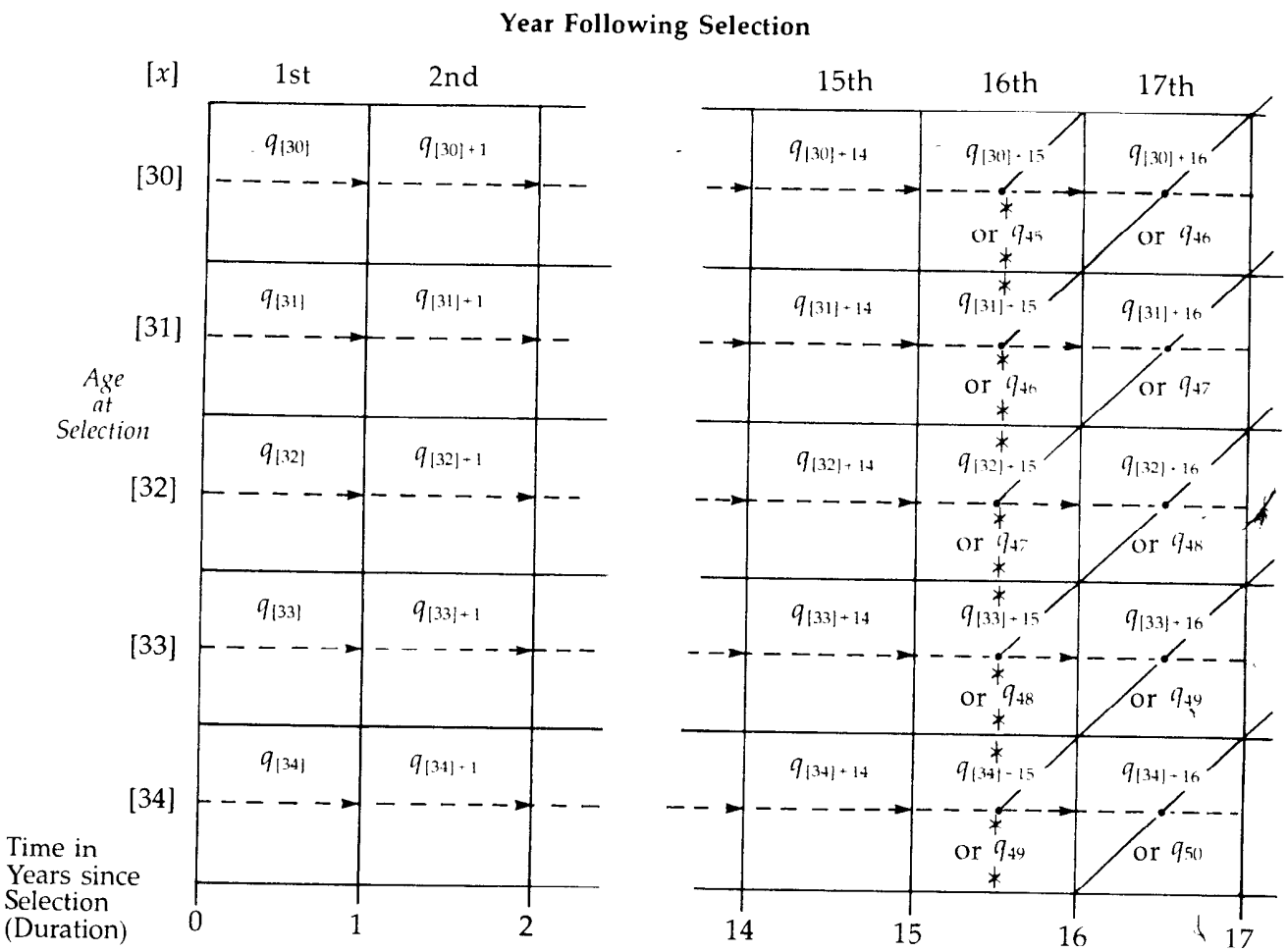
The calculations of the basic functions q_x, l_x, and d_x from (3.7.1) were all done directly from (3.7.1) instead of calculating l_x and d_x from the truncated values of q_x. It was found that the latter choice would make little difference in the applications. It should be kept in mind that the Illustrative Life Table, as its name implies, is for illustrative purposes only.

3.8 Select and Ultimate Tables

In Section 3.2 we discussed how ${}_tp_x$ [the probability that (x) will survive to age $x + t$] might be interpreted in two ways. The first interpretation was that the probability can be evaluated by a survival function appropriate for newborns, under the single hypothesis that the newborn has survived to age x. This interpretation has been the basis of the notation and development of the formulas. The second interpretation was that additional knowledge available about the life at age x might make the original survival function inappropriate for evaluating probability statements about the future lifetime of (x). For example, the life might have been underwritten and accepted for life insurance at age x. This information would lead us to believe that (x)'s future-lifetime distribution is different from what we might otherwise assume for lives age x. As a second example, the life might have become disabled at age x. This information would lead us to believe that the future-lifetime distribution for (x) is different from that of those not disabled at age x. In these two illustrations, a special force of mortality that incorporates the particular information available at age x would be preferred. Without this particular information for (x), the form of mortality at duration t would be a function of only the attained age $x + t$, denoted in the previous sections by $\mu(x + t)$. Given the additional information at x, the force of mortality at $x + t$ is a function of this information at x and duration t. Its notation will be $\mu_x(t)$, showing separately the age, x, at which the additional information was available, and the duration, t. The additional information is usually not explicit in the notation but is conveyed by the context. In other words, the complete model for such lives is a set of survival functions including one for each age at which information is available on issue of insurance, disability, and so on. This set of survival functions can be thought of as a function of two variables. One variable is the age at ***selection*** (e.g., at policy issue or the onset of disability), $[x]$, and the second variable is the duration since policy issue or duration since selection, t. Then each of the usual life table functions associated with this bivariate survival function is a two-dimensional array on $[x]$ and t. Note the bracket notation to indicate which variable identifies the age of selection.

The schematic diagram in Figure 3.8.1 illustrates these ideas. For instance, suppose some special information is available about a group of lives age 30. Perhaps they have been accepted for life insurance or perhaps they have become disabled. A special life table can be built for these lives. The conditional probability of death in each year of duration would be denoted by $q_{[30]+i}$, $i = 0, 1, 2, \ldots$, and would be entered on the first row of Figure 3.8.1. The subscript reflects the bivariate nature of this function with the bracketed thirty, [30], denoting that the survival function in the first row is conditional on special information available at age 30. The second row of Figure 3.8.1 would contain the probabilities of death for lives on which the

Select, Ultimate, and Aggregate Mortality, 15-Year Select Period



Path followed by a survivorship group selected at $[x]$.

Links (age at selection and duration) cells after a 15-year select period into attained age groups.

Alternative path followed by survivorship groups after a 15-year select period. These probabilities constitute an ultimate mortality table.

Notes:

1. In biostatistics the select table index $[x]$ may not be age. For example, in cancer research, $[x]$ could be a classification index that depends on the size and location of the tumor, and time following selection would be measured from the time of diagnosis.
2. Ultimate mortality, following a 15-year select period, for age $[x] + 15$, would be estimated by using observations from all cells identified by $[x-j] + 15 + j$, for $j = 0, 1, 2, \ldots$. Therefore, $q_{[x]+15} = q_{x+15}$ is estimated by a weighted average of mortality estimates from several different selection groups. If the effect of selection is not small, the resulting estimate will be influenced by the amount of data from the various cells.

special information became available at age 31. In actuarial science such a two-dimensional life table is called a ***select life*** table.

The impact of selection on the distribution of time-until-death, T, may diminish following selection. Beyond this time period the q's at equal attained ages would be essentially equal regardless of the ages at selection. More precisely, if there is a smallest integer r such that $|q_{[x]+r} - q_{[x-j]+r+j}|$ is less than some small positive constant for all ages of selection $[x]$ and for all $j > 0$, it would be economical to construct a set of ***select-and-ultimate*** tables by truncation of the two-dimensional array after the $(r + 1)$ column. For durations beyond r we would use

$$q_{[x-j]+r+j} \cong q_{[x]+r} \qquad j > 0.$$

The first r years of duration comprise the ***select period.***

The resulting array remains a set of life tables, one for each age at selection. For a single age at selection, the life table entries are horizontal during the select period and then vertical during the ultimate period. This is shown in Figure 3.8.1 by the arrows.

The Society of Actuaries mortality studies of lives who were issued individual life insurance on a standard basis use a 15-year select period as illustrated in Figure 3.8.1; that is, it is accepted that

$$q_{[x-j]+15+j} \cong q_{[x]+15} \qquad j > 0.$$

Beyond the select period, the probabilities of death are subscripted by attained age only; that is, $q_{[x-j]+r+j}$ is written as q_{x+r}. For instance, with $r = 15$, $q_{[30]+15}$ and $q_{[25]+20}$ would both be written as q_{45}.

A life table in which the functions are given only for attained ages is called an ***aggregate table,*** Table 3.3.1, for instance. The last column in a select-and-ultimate table is a special aggregate table that is usually referred to as an ***ultimate table,*** to reflect the select table setting.

Table 3.8.1 contains mortality probabilities and corresponding values of the $l_{[x]+k}$ function, as given in the Permanent Assurances, Females, 1979–82, Table, published by the Institute of Actuaries and the Faculty of Actuaries; it is denoted as the

Excerpt from the AF80 Select-and-Ultimate Table

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$[x]$	$1{,}000\ q_{[x]}$	$1{,}000\ q_{[x]+1}$	$1{,}000\ q_{x+2}$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x + 2$
30	0.222	0.330	0.422	9 906.7380	9 904.5387	9 901.2702	32
31	0.234	0.352	0.459	9 902.8941	9 900.5769	9 897.0919	33
32	0.250	0.377	0.500	9 898.7547	9 896.2800	9 892.5491	34
33	0.269	0.407	0.545	9 894.2903	9 891.6287	9 887.6028	35
34	0.291	0.441	0.596	9 889.4519	9 886.5741	9 882.2141	36

AF80 Table. This table has a 2-year select period and is easier to use for illustrative purposes than tables with a 15-year select period such as the Basic Tables, published by the Society of Actuaries.

In Table 3.8.1 we observe three mortality probabilities for age 32, namely,

$$q_{[32]} = 0.000250 < q_{[31]+1} = 0.000352 < q_{32} = 0.000422.$$

The order among these probabilities is plausible since mortality should be lower for lives immediately after acceptance for life insurance. Column (3) can be viewed as providing ultimate mortality probabilities.

Given the 1-year mortality rates of a select-and-ultimate table, the construction of the corresponding select-and-ultimate life table (survival functions) is started with the ultimate portion. Formulas such as (3.4.1) can be used, which would yield a set of values of $l_{x+r} = l_{[x]+r}$ where r is the length of the select period. We would then complete the select segments by using the relation

$$l_{[x]+r-k-1} = \frac{l_{[x]+r-k}}{p_{[x]+r-k-1}} \qquad k = 0, 1, 2, \ldots, r-1,$$

working from duration $r - 1$ down to 0.

Example 3.8.1

Use Table 3.8.1 to evaluate

a. ${}_2p_{[30]}$ b. ${}_5p_{[30]}$

c. ${}_{1|}q_{[31]}$ d. ${}_3q_{[31]+1}$.

Solution:

Formulas developed earlier in this chapter can be adapted to select-and-ultimate tables yielding

a. ${}_2p_{[30]} = \dfrac{l_{[30]+2}}{l_{[30]}} = \dfrac{l_{32}}{l_{[30]}} = \dfrac{9{,}901.2702}{9{,}906.7380} = 0.99945$

b. ${}_5p_{[30]} = \dfrac{l_{35}}{l_{[30]}} = \dfrac{9{,}887.6028}{9{,}906.7380} = 0.99807$

c. ${}_{1|}q_{[31]} = \dfrac{l_{[31]+1} - l_{33}}{l_{[31]}} = \dfrac{9{,}900.5769 - 9{,}897.0919}{9{,}902.8941} = 0.00035$

d. ${}_3q_{[31]+1} = \dfrac{l_{[31]+1} - l_{35}}{l_{[31]+1}} = \dfrac{9{,}900.5769 - 9{,}887.6028}{9{,}900.5769} = 0.00131.$ ▼

Chapter 3 Concepts

Symbol	Name or Description of the Concept
(x)	Notation for a life age x
$[x]$	Age, or other status, at selection
X	Age at death, a random variable
$T(x)$	Future lifetime of (x), equals $X - x$
$K(x)$	Curtate-future-lifetime of (x), equals the integer part of $T(x)$
$S(x)$	Future lifetime of (x) within the year of death, equals $T(x) - K(x)$
$s(x)$	Survival function, equal to the probability that a newborn will live to at least x
$\mu(x)$	Force of mortality at age x in an aggregate life table
$\mu_x(t)$	Force of mortality at attained age $x + t$ given selection at age x
${}_tq_x$	Probability that (x) dies within t years
${}_tp_x$	Probability that (x) survives at least t years
${}_{t\|u}q_x$	Probability that (x) dies between t and $t + u$ years
$\mathring{e}_x$	Complete expectation of life for (x), equals $E[T(x)]$
e_x	Curtate expectation of life for (x), equals $E[K(x)]$
$\mathcal{L}(x)$	Cohort's number of survivors to age x, a random variable
${}_n\mathcal{L}_x$	Cohort's number of deaths between ages x and $x + n$
l_x	Expected number of survivors at age x, equals $E[\mathcal{L}(x)]$
${}_nd_x$	Expected number of deaths between ages x and $x + n$, equals $E[{}_n\mathcal{L}_x]$
${}_nL_x$	Expected number of years lived between ages x and $x + n$ by survivors to age x of the initial group of l_0 lives
T_x	Expected number of years lived beyond age x by the survivors to age x of the initial group of l_0 lives
m_x	Central death rate over the interval $(x, x + 1)$
ω	Omega, the limiting age of a life table

3.9 Notes and References

Table 3.9.1 summarizes this chapter's new concepts with their names, symbols, and descriptions. Life tables are a cornerstone of actuarial science. Consequently they are extensively discussed in several English-language textbooks on life contingencies:

- King (1902)
- Spurgeon (1932)
- Jordan (1967)
- Hooker and Longley-Cook (1953)
- Neill (1977).

These have been used in actuarial education. In addition, life tables are used by biostatisticians. An exposition of this latter approach is given by Chiang (1968) and Elandt-Johnson and Johnson (1980). The deterministic rate function interpretation is discussed by Allen (1907). London (1988) summarizes several methods for estimating life tables from data.

The historically important analytic forms for survival functions are referred to in Table 3.6. Brillinger (1961) provides an argument for certain analytic forms from the viewpoint of statistical life testing. Tenenbein and Vanderhoof (1980) restate the case for analytic laws of mortality and develop formulas for select mortality. Balducci's (1921) contribution was preceded by a remarkable set of papers by Wittstein (1873). Wittstein's papers were published first in German and translated into English by T. B. Sprague. Some of the methods for evaluating probabilities for fractional ages are reviewed by Mereu (1961) and in Batten's textbook on mortality estimation (1978) (see also Seal's 1977 historical review). Discussions of the length of the select period for various types of insurance selection procedures have a long history, for example, Williamson (1942), Thompson (1934), and Jenkins (1943). The Society of Actuaries 1975–80 Basic Tables use a 15-year select period and are published in *TSA Reports 1982*. International Actuarial Notation is outlined in *TASA 48* (1947).

We planned to use the 1989–91 U.S. Life Table for illustrative purposes in Table 3.2.1, but this plan was not realized because the life tables based on the 1990 U.S. Census were not completed when this chapter was revised.

Exercises

Section 3.2

3.1. Using the ideas summarized in Table 3.2.1, complete the entries below.

$s(x)$	$F_X(x)$	$f_X(x)$	$\mu(x)$
			$\tan x,\ 0 \le x \le \frac{\pi}{2}$
$e^{-x},\ x \ge 0$			
	$1 - \frac{1}{1+x},\ x \ge 0$		

3.2. Confirm that each of the following functions can serve as a force of mortality. Show the corresponding survival function. In each case $x \ge 0$.

a. $B\,c^x$ $\quad B > 0 \quad c > 1$ (Gompertz)

b. $k\,x^n$ $\quad n > 0 \quad k > 0$ (Weibull)

c. $a\,(b + x)^{-1}$ $\quad a > 0 \quad b > 0$ (Pareto)

3.3. Confirm that the following can serve as a survival function. Show the corresponding $\mu(x)$, $f_X(x)$, and $F_X(x)$.

$$s(x) = e^{-x^3/12} \qquad x \ge 0.$$

3.4. State why each of the following functions cannot serve in the role indicated by the symbol:

a. $\mu(x) = (1 + x)^{-3} \qquad x \ge 0$

b. $s(x) = 1 - \frac{22x}{12} + \frac{11x^2}{8} - \frac{7x^3}{24} \qquad 0 \le x \le 3$

c. $f_X(x) = x^{n-1} e^{-x/2} \qquad x \ge 0,\ n \ge 1.$

3.5. If $s(x) = 1 - x/100$, $0 \le x \le 100$, calculate

a. $\mu(x)$ b. $F_X(x)$

c. $f_X(x)$ d. Pr $(10 < X < 40)$.

3.6. Given the survival function of Exercise 3.5, determine the survival function, force of mortality, and p.d.f. of the future lifetime of (40).

3.7. If $s(x) = [1 - (x/100)]^{1/2}$, $0 \le x \le 100$, evaluate

a. ${}_{17}p_{19}$ b. ${}_{15}q_{36}$ c. ${}_{15|13}q_{36}$

d. $\mu(36)$ e. $E[T(36)]$.

3.8. Confirm that ${}_{k|}q_0 = -\Delta s(k)$, and that $\sum_{k=0}^{\infty} {}_{k|}q_0 = 1$.

3.9. If $\mu(x) = 0.001$ for $20 \le x \le 25$, evaluate ${}_{2|2}q_{20}$.

Sections 3.3, 3.4

3.10. If the survival times of 10 lives in a survivorship group are independent with survival defined in Table 3.3.1, exhibit the p.f. of $\mathcal{L}(65)$ and the mean and variance of $\mathcal{L}(65)$.

3.11. If $s(x) = 1 - x/12$, $0 \le x \le 12$, $l_0 = 9$, and the survival times are independent, then $({}_3\mathcal{Y}_0, {}_3\mathcal{Y}_3, {}_3\mathcal{Y}_6, {}_3\mathcal{Y}_9)$ is known to have a multinomial distribution. Calculate

a. The expected value of each random variable

b. The variance of each random variable

c. The coefficient of correlation between each pair of random variables.

3.12. On the basis of Table 3.3.1,

a. Compare the values of ${}_5q_0$ and ${}_5q_5$

b. Evaluate the probability that (25) will die between ages 80 and 85.

3.13. Given that l_{x+t} is strictly decreasing in the interval $0 \le t \le 1$, show that

a. If l_{x+t} is concave down, then $q_x > \mu(x)$

b. If l_{x+t} is concave up, then $q_x < \mu(x)$.

3.14. Show that

a. $\frac{d}{dx} l_x\, \mu(x) < 0$ when $\frac{d}{dx} \mu(x) < \mu^2(x)$

b. $\frac{d}{dx} l_x\, \mu(x) = 0$ when $\frac{d}{dx} \mu(x) = \mu^2(x)$

c. $\frac{d}{dx} l_x\, \mu(x) > 0$ when $\frac{d}{dx} \mu(x) > \mu^2(x)$.

3.15. Consider a random survivorship group consisting of two subgroups: (1) the survivors of 1,600 persons joining at birth; (2) the survivors of 540 persons joining at age 10. An excerpt from the appropriate mortality table for both subgroups follows:

x	l_x
0	40
10	39
70	26

If Y_1 and Y_2 are the numbers of survivors to age 70 out of subgroups (1) and (2), respectively, estimate a number c such that $\Pr(Y_1 + Y_2 > c) = 0.05$. Assume the lives are independent and ignore half-unit corrections.

Section 3.5

3.16. Let the random variable

$$T^*(x) = T(x) \qquad 0 < T(x) \leq n$$
$$= n \qquad n < T(x)$$

and denote $\mathrm{E}[T^*(x)]$ by $\mathring{e}_{x:\overline{n}|}$. This expectation is called a ***temporary complete life expectancy.*** It is used in public health planning; the same expectation, under the name ***limited expected value function,*** is used in the analysis of loss amount distributions. Show that

a. $\mathring{e}_{x:\overline{n}|} = \int_0^n t\,{}_tp_x\,\mu(x+t)\,dt + n\,{}_np_x$

$$= \int_0^n {}_tp_x\,dt = \frac{T_x - T_{x+n}}{l_x}$$

b. $\mathrm{Var}[T^*(x)] = \int_0^n t^2\,{}_tp_x\,\mu(x+t)\,dt + n^2\,{}_np_x - (\mathring{e}_{x:\overline{n}|})^2$

$$= 2\int_0^n t\,{}_tp_x\,dt - \mathring{e}_{x:\overline{n}|}^2.$$

3.17. Let the random variable

$$K^*(x) = K(x) \qquad K(x) = 0, 1, 2, \ldots, n-1$$
$$= n \qquad K(x) = n, n+1, \ldots$$

and denote $\mathrm{E}[K^*(x)]$ by $e_{x:\overline{n}|}$. This expectation is called a ***temporary curtate life expectancy.*** Show that

a. $e_{x:\overline{n}|} = \sum_0^{n-1} k\,{}_k|q_x + n\,{}_np_x$

$$= \sum_1^n {}_kp_x$$

b. $\text{Var}[K^*(x)] = \sum_{0}^{n-1} k^2 \,{}_{k|}q_x + n^2 \,{}_{n}p_x - (e_{x:\overline{n}|})^2$

$= \sum_{1}^{n} (2k+1)\,{}_{k}p_x - (e_{x:\overline{n}|})^2.$

3.18. If the random variable T has p.d.f. given by $f_T(t) = ce^{-ct}$ for $t \geq 0$, $c > 0$, calculate

a. $\mathring{e}_x = \text{E}[T]$ b. $\text{Var}(T)$ c. median (T)

d. The mode of the distribution of T.

3.19. If $\mu(x+t) = t$, $t \geq 0$, calculate

a. ${}_tp_x\,\mu(x+t)$ b. $\mathring{e}_x$.

[Hint: Recall, from the study of probability, that $(1/\sqrt{2\pi})\,e^{-t^2/2}$ is the p.d.f. for the standard normal distribution.]

3.20. If the random variable $T(x)$ has d.f. given by

$$F_{T(x)}(t) = \begin{cases} \dfrac{t}{(100-x)} & 0 \leq t < 100 - x \\ 1 & t \geq 100 - x, \end{cases}$$

calculate

a. $\mathring{e}_x$ b. $\text{Var}[T(x)]$ c. median $[T(x)]$.

3.21. Show that

a. $\dfrac{\partial}{\partial x}\,{}_tp_x = {}_tp_x\,[\mu(x) - \mu(x+t)]$

b. $\dfrac{d}{dx}\,\mathring{e}_x = \mathring{e}_x\,\mu(x) - 1$

c. $\Delta e_x = q_x\,e_{x+1} - p_x$.

3.22. Confirm the following statements:

a. $a(x)\,d_x = L_x - l_{x+1}$

b. The approximation developed in Example 3.5.1 was not used to calculate L_0 in Table 3.3.1, but was used to calculate L_1

c. $T_x = \sum_{k=0}^{\infty} L_{x+k}$.

3.23. The survival function is given by

$$s(x) = 1 - \frac{x}{10} \qquad 0 \leq x \leq 10$$

$$= 0 \qquad \text{elsewhere.}$$

Calculate values of $\mathring{e}_x$ and e_x, $x = 0, 1, 2, \ldots, 9$

a. Using formulas (3.5.2) and (3.5.7)

b. Using the formulas developed in Table 3.5.1.

3.24. Find $u(0)$, $-c(x)/d(x)$, and $d(x)$ if $u(x) = \Pr[X = x]$ where (3.5.20) is to be used to produce a table of the p.f. of the random variable X when it has a
   a. A Poisson distribution with parameter λ
   b. A binomial distribution with parameters n and p.

3.25. Formula (3.5.20) is to be used to produce tables of compound interest functions. Find $u(1)$, $-c(x)/d(x)$, and $1/d(x)$ when
   a. $u(x) = \ddot{a}_{\overline{x}|}$
   b. $u(x) = \ddot{s}_{\overline{x}|}$.

Section 3.6

3.26. Verify the entries for the constant force of mortality and the hyperbolic assumption in Table 3.6.1. Note that the entry for ${}_tp_x$ in the hyperbolic column provides justification for the hyperbolic name.

3.27. Graph $\mu(x + t)$, $0 < t < 1$, for each of the three assumptions in Table 3.6.1. Also graph the survival function for each assumption.

3.28. Using the l_x column of Table 3.3.1, compute ${}_{1/2}p_{65}$ for each of the three assumptions in Table 3.6.1.

3.29. Use Table 3.3.1 and an assumption of uniform distribution of deaths in each year of age to find the median of the future lifetime of a person
   a. Age 0 b. Age 50.

3.30. If $q_{70} = 0.04$ and $q_{71} = 0.05$, calculate the probability that (70) will die between ages 70½ and 71½ under
   a. The assumption that deaths are uniformly distributed within each year of age
   b. The hyperbolic assumption for each year of age.

3.31. Using the l_x column in Table 3.3.1 and each of the assumptions in Table 3.6.1, compute
   a. $\lim_{h \to 0} \mu(60 + h)$
   b. $\lim_{h \to 0^-} \mu(60 + h)$
   c. $\mu(60 + \frac{1}{2})$.

3.32. If the constant force assumption is adopted, show that
   a. $a(x) = \dfrac{[(1 - e^{-\mu})/\mu] - e^{-\mu}}{1 - e^{-\mu}}$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{12}$.

3.33. If the hyperbolic assumption is adopted, show
   a. $a(x) = -\dfrac{p_x}{q_x^2}(q_x + \log p_x)$ b. $a(x) \cong \dfrac{1}{2} - \dfrac{q_x}{6}$.

Section 3.7

3.34. Verify the entries in Table 3.7.1 for De Moivre's law and Weibull's law.

3.35. Consider a modification of De Moivre's law given by

$$s(x) = \left(1 - \frac{x}{\omega}\right)^{\alpha} \qquad 0 \le x < \omega, \qquad \alpha > 0.$$

Calculate

a. $\mu(x)$ b. $\mathring{e}_x$.

Section 3.8

3.36. Using Table 3.8.1, calculate

a. ${}_2q_{[32]+1}$ b. ${}_2p_{[31]+1}$.

3.37. The quantity

$$1 - \frac{q_{[x]+k}}{q_{x+k}} = I(x, k)$$

has been called the ***index of selection.*** When it is close to 0, the indication is that selection has worn off. From Table 3.8.1, calculate the index for $x = 32$, $k = 0, 1$.

3.38. The force of mortality for a life selected at age (x) is given by $\mu_{\mathbf{x}}(t) = \Psi(\mathbf{x})\mu(t)$, $t > 0$. In this formula $\mu(t)$ is the standard force of mortality. The symbol $\mathbf{x}$ denotes a vector of numerical information about the life at the time of selection. This information would include the age and other classification information. It is required that $\Psi(\mathbf{x}) > 0$ and $\Psi(\mathbf{x}_0) = 1$, where $\mathbf{x}_0$ denotes standard information. Show that the select survival function is

$${}_tp_{[\mathbf{x}_0]} = ({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})}$$

and the p.d.f. of $T(\mathbf{x})$, the random variable time-until-death given the information x, is $-\Psi(\mathbf{x})\ {}_tp'_{[\mathbf{x}_0]}({}_tp_{[\mathbf{x}_0]})^{\Psi(\mathbf{x})-1}$, where ${}_tp'_{[\mathbf{x}_0]}$ is the derivative with respect to t of ${}_tp_{[\mathbf{x}_0]}$. This is called a ***proportional hazard model.***

Miscellaneous

3.39. A life at age 50 is subject to an extra hazard during the year of age 50 to 51. If the standard probability of death from age 50 to 51 is 0.006, and if the extra risk may be expressed by an addition to the standard force of mortality that decreases uniformly from 0.03 at the beginning of year to 0 at the end of the year, calculate the probability that the life will survive to age 51.

3.40. If the force of mortality $\mu_x(t)$, $0 \le t \le 1$, changes to $\mu_x(t) - c$ where c is a positive constant, find the value of c for which the probability that (x) will die within a year will be halved. Express the answer in terms of $q_{[x]}$.

3.41. From a standard mortality table, a second table is prepared by doubling the force of mortality of the standard table. Is the rate of mortality, q'_x, at any

given age under the new table, more than double, exactly double, or less than double the mortality rate, q_x, of the standard table?

3.42. If $\mu(x) = B\,c^x$, $c > 1$, show that the function $l_x\,\mu(x)$ has its maximum at age x_0 where $\mu(x_0) = \log c$. [Hint: This exercise makes use of Exercise 3.14.]

3.43. Assume $\mu(x) = \dfrac{A\,c^x}{1 + B\,c^x}$ for $x > 0$.

a. Calculate the survival function, $s(x)$.

b. Verify that the mode of the distribution of X, the age-at-death, is given by

$$x_0 = \frac{\log(\log c) - \log A}{\log c}.$$

3.44. If $\mu(x) = \dfrac{3}{100 - x} - \dfrac{10}{250 - x}$ for $40 < x < 100$, calculate

a. ${}_{40}p_{50}$

b. The mode of the distribution of X, the age-at-death.

3.45. a. Show that, under the uniform distribution of deaths assumption,

$$m_x = \frac{q_x}{1 - (1/2)q_x} \quad \text{and} \quad q_x = \frac{m_x}{1 + (1/2)m_x}.$$

b. Calculate m_x in terms of q_x under the constant force assumption.

c. Calculate m_x in terms of q_x under the hyperbolic assumption.

d. If $l_x = 100 - x$ for $0 \le x \le 100$, calculate ${}_{10}m_{50}$ where

$${}_nm_x = \frac{\displaystyle\int_0^n l_{x+t}\,\mu(x + t)\,dt}{\displaystyle\int_0^n l_{x+t}\,dt}.$$

3.46. Show that K and S are independent if and only if the expression

$$\frac{{}_sq_{[x]+k}}{q_{[x]+k}}$$

does not depend on k for $0 \le s \le 1$.

Computing Exercises:

These are the first in a series of exercises that involve sufficient computation to make it worthwhile to use a computer. The series will continue in the following chapters, and in each exercise it is assumed that the results of previous exercises are available. For example, in Exercise 3.47, you are asked to set up a life table that will then be used in risk analysis in Chapters 4 and 5.

3.47. Using spreadsheet or other mathematical software, set up an object that will accept input values for the Makeham law parameters and then calculate

and display the values of p_x and q_x for ages 0 to 140. As a check on your output, input the parameter values given in (3.7.1) and compare your q_x values with those for $x = 13, 14, \ldots$ in the Illustrative Life Table in Appendix 2A. We will refer to this computing object as your Illustrative Life Table. When the Makeham parameter values are not stated, those of (3.7.1) are implied. [Remark: With a Makeham Table, $s(x) > 0$ for all $x > 0$, so ω does not exist as defined in Section 3.3.1. For the parameter values of the Illustrative Life Table, q_{140} is zero to eight decimal places; thus we choose $\omega = 140$ for our Illustrative Life Table, that is, Table 2A.]

3.48. In your Illustrative Life Table use the forward recursion formula $l_{x+1} = (p_x)(l_x)$ and initial value $l_{13} = 96{,}807.88$ to calculate the l_x values of Table 2A. [Remark: The Makeham law was not realistic for ages less than 13, so the Illustrative Life Table is a blend of some ad hoc values from 0 through 12 and the Makeham law table from age 13 up.]

3.49. Illustrate the result of Exercise 3.41 by doubling the A and B parameter values in your Illustrative Life Table.

3.50. Use the backward recursion formula of Table 3.5.1 to calculate values of e_x in your Illustrative Life Table for ages 13 to 140.

3.51. Compare the values of e_x at $x = 20, 40, 60, 80$, and 100 in your Illustrative Life Table with those found when the force of mortality is doubled.

3.52. Use the backward recursion formula of Table 3.5.1 and the trapezoidal rule to calculate values of $\mathring{e}_x$ in your Illustrative Life Table for ages 13 to 110.

3.53. Verify the following backward recursion formula for the temporary curtate life expectancy to age y:

$$e_{x:\overline{y-x}|} = p_x + p_x\, e_{x+1:\overline{y-(x+1)}|} \quad \text{for } x = 0, 1, \ldots, y-1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the curtate temporary life expectancy up to age 45 for ages 13 to 44.

3.54. Verify the following backward recursion formula for the n-year temporary curtate life expectancy:

$$e_{x:\overline{n}|} = p_x\,(1 - {}_np_{x+1}) + p_x\, e_{x+1:\overline{n}|} \quad \text{for } x = 0, 1, \ldots, \omega - 1.$$

Determine an appropriate starting value for use with this formula. For your Illustrative Life Table calculate the 10-year temporary curtate life expectancy for ages 13 to 139.

3.55. "Look up" $e_{15:\overline{25}|}$ in your Illustrative Life Table. [Hint: Since the $c(x)$ term in the relation in Exercise 3.53 does not depend on n, it may be more efficient to view $e_{15:\overline{25}|}$ as a curtate temporary life expectancy to age 40 for (15).]





LIFE INSURANCE

4.1 Introduction

We have stated that insurance systems are established to reduce the adverse financial impact of some types of random events. Within these systems individuals and organizations adopt utility models to represent preferences, stochastic models to represent uncertain financial impact, and economic principles to guide pricing. Agreements are reached after analyses of these models.

In Chapter 2 we developed an elementary model for the financial impact of random events in which the occurrence and the size of impact are both uncertain. In that model, the policy term is assumed to be sufficiently short so the uncertainty of investment income from a random payment time could be ignored.

In this chapter we develop models for life insurances designed to reduce the financial impact of the random event of untimely death. Due to the long-term nature of these insurances, the amount of investment earnings, up to the time of payment, provides a significant element of uncertainty. This uncertainty has two causes: the unknown rate of earnings over, and the unknown length of, the investment period. A probability distribution is used to model the uncertainty in regards to the investment period throughout this book. In this chapter a deterministic model is used for the unknown investment earnings, and in Chapter 21 stochastic models for this uncertainty are discussed. In other words, our model will be built in terms of functions of T, the insured's future-lifetime random variable.

While everything in this chapter will be stated as insurances on human lives, the ideas would be the same for other objects such as equipment, machines, loans, and business ventures. In fact, the general model is useful in any situation where the size and time of a financial impact can be expressed solely in terms of the time of the random event.

4.2 Insurances Payable at the Moment of Death

In this chapter, the amount and the time of payment of a life insurance benefit depend only on the length of the interval from the issue of the insurance to the death of the insured. Our model will be developed with a ***benefit function,*** b_t, and a ***discount function,*** v_t. In our model, v_t is the interest discount factor from the time of payment back to the time of policy issue, and t is the length of the interval from issue to death. In the case of endowments, covered in this section, t can be greater than or equal to the length of the interval from issue to payment.

For the discount function we assume that the underlying force of interest is deterministic; that is, the model does not include a probability distribution for the force of interest. Moreover, we usually show the simple formulas resulting from the assumption of a constant, as well as a deterministic, force of interest.

We define the ***present-value function,*** z_t, by

$$z_t = b_t v_t. \tag{4.2.1}$$

Thus, z_t is the present value, at policy issue, of the benefit payment. The elapsed time from policy issue to the death of the insured is the insured's future-lifetime random variable, $T = T(x)$, defined in Section 3.2.2. Thus, the present value, at policy issue, of the benefit payment is the random variable z_T. Unless the context requires a more elaborate symbol, we denote this random variable by Z and base the model for the insurance on the equation

$$Z = b_T v_T. \tag{4.2.2}$$

The random variable Z is an example of a claim random variable and, as such, of an X_t term in the sum of the individual risk model, as defined by (2.1.1). This model is used in later sections when we consider applications involving portfolios. We now turn to the development of the probability model for Z.

The first step in our analysis of a life insurance will be to define b_t and v_t. The next step is to determine some characteristics of the probability distribution of Z that are consequences of an assumed distribution for T, and we work through these steps for several conventional insurances. A summary is provided in Table 4.2.1 on page 109.

4.2.1 Level Benefit Insurance

An ***n-year term life insurance*** provides for a payment only if the insured dies within the n-year term of an insurance commencing at issue. If a unit is payable at the moment of death of (x), then

$$b_t = \begin{cases} 1 & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t \ge 0,$$

$$Z = \begin{cases} v^T & T \le n \\ 0 & T > n. \end{cases}$$

These definitions use three conventions. First, since the future lifetime is a non-negative variable, we define b_t, v_t, and Z only on non-negative values. Second, for a t value where b_t is 0, the value of v_t is irrelevant. At these values of t, we adopt definitions of v_t by convenience. Third, unless stated otherwise, the force of interest is assumed to be constant.

The expectation of the present-value random variable, Z, is called the ***actuarial present value*** of the insurance. The reader will find that the expectation of the present value of a set of payments contingent on the occurrence of a set of events is referred to by different terms in different actuarial contexts. In Chapter 1, the expected loss was called the pure premium. This vocabulary is commonly used in property-liability insurance. A more exact term, but more cumbersome, would be ***expectation of the present value of the payments.*** We denote actuarial present values by their symbols according to the International Actuarial Notation (see Appendix 4).

The principal symbol for the actuarial present value of an insurance paying a unit benefit is A. The subscript includes the age of the insured life at the time of the calculation. How this age is displayed depends upon the form of the mortality assumption. For the actuarial present value of an insurance on (40), the age might be displayed as [40], 40, or [20] + 20, for example. As in Section 3.8, the bracket indicates selection at that age and hence the use of a select table commencing at that age. The unbracketed age indicates the use of an aggregate or ultimate table. Thus [20] + 20 indicates the calculation for a 40-year-old on the basis of a select table commencing at age 20.

The actuarial present value for the n-year term insurance with a unit payable at the moment of death of (x), E[Z], is denoted by $\bar{A}^1_{x:\overline{n}|}$. This can be calculated by recognizing Z as a function of T so that $E[Z] = E[z_T]$. Then we use the p.d.f. of T to obtain

$$\bar{A}^1_{x:\overline{n}|} = E[Z] = E[z_T] = \int_0^\infty z_t \, f_T(t) \, dt = \int_0^n v^t \, {}_tp_x \, \mu_x(t) \, dt. \qquad (4.2.3)$$

The j-th moment of the distribution of Z can be found by

$$E[Z^j] = \int_0^n (v^t)^j \, {}_tp_x \, \mu_x(t) \, dt$$

$$= \int_0^n e^{-(\delta_t)t} \, {}_tp_x \, \mu_x(t) \, dt.$$

- The second integral shows that the j-th moment of Z is equal to the actuarial present value for an n-year term insurance for a unit amount payable at the moment of death of (x), calculated at a force of interest equal to j times the given force of interest, or $j\delta$.

This property, which we call the ***rule of moments,*** holds generally for insurances paying only a unit amount when the force of interest is deterministic, constant or not. More precisely,

$$\mathrm{E}[Z^j] \,@\, \delta_t = \mathrm{E}[Z] \,@\, j\delta_t. \tag{4.2.4}$$

In addition to the existence of the moments, the sufficient condition for the rule of moments is $b_t^j = b_t$ for all $t \geq 0$, that is, for each t the benefit amount is 0 or 1. Demonstration that this is sufficient is left to Exercise 4.30.

It follows from the rule of moments that

$$\mathrm{Var}(Z) = {}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2 \tag{4.2.5}$$

where ${}^2\bar{A}^1_{x:\overline{n}|}$ is the actuarial present value for an n-year term insurance for a unit amount calculated at force of interest 2δ.

Whole life insurance provides for a payment following the death of the insured at any time in the future. If the payment is to be a unit amount at the moment of death of (x), then

$$\begin{aligned} b_t &= 1 \qquad t \geq 0, \\ v_t &= v^t \qquad t \geq 0, \\ Z &= v^T \qquad T \geq 0. \end{aligned}$$

The actuarial present value is

$$\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.6}$$

For a life selected at x and now age $x + h$, the expression would be

$$\bar{A}_{[x]+h} = \int_0^\infty v^t \, {}_tp_{[x]+h} \, \mu_x(h + t) \, dt.$$

Whole life insurance is the limiting case of n-year term insurance as $n \to \infty$.

Example 4.2.1

The p.d.f. of the future lifetime, T, for (x) is assumed to be

$$f_T(t) = \begin{cases} 1/80 & 0 \leq t \leq 80 \\ 0 & \text{elsewhere.} \end{cases}$$

At a force of interest, δ, calculate for Z, the present-value random variable for a whole life insurance of unit amount issued to (x):

a. The actuarial present value

b. The variance

c. The 90th percentile, $\xi_Z^{0.9}$.

Solution:

a. $\bar{A}_x = \mathrm{E}[Z] = \int_0^\infty v^t f_T(t)\, dt = \int_0^{80} e^{-\delta t} \frac{1}{80}\, dt = \frac{1 - e^{-80\delta}}{80\,\delta} \qquad \delta \neq 0.$

b. By the rule of moments,

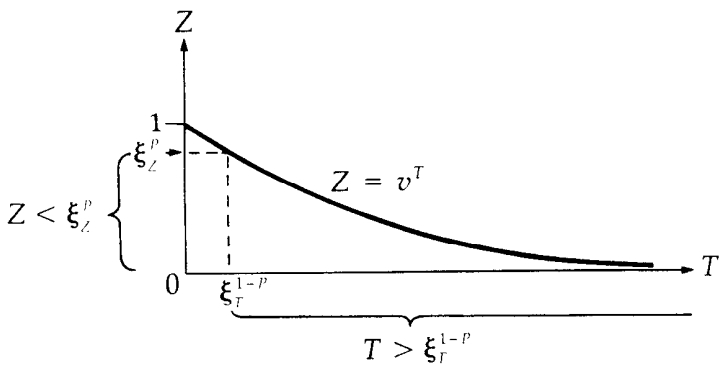
$$\mathrm{Var}(Z) = \frac{1 - e^{-160\delta}}{160\,\delta} - \left(\frac{1 - e^{-80\delta}}{80\,\delta}\right)^2 \qquad \delta \neq 0.$$

c. For the continuous random variable, Z, we have $\Pr(Z \leq \xi_Z^{0.9}) = 0.9$.

Since we have the p.d.f. for T and not for Z, we proceed by finding the event for T which corresponds to $Z \leq \xi_Z^{0.9}$. From Figure 4.2.1, which shows the general relationship between the sample space of T (on the horizontal axis) and the sample space of Z (on the vertical axis), we see that $\xi_Z^{0.9} = v^{\xi_T^{0.1}}$. Because Z is a strictly decreasing function of T for whole life insurance, the percentile from T's distribution that is related to 90th percentile of Z's distribution is at the complementary probability level, 0.1. In this example T is uniformly distributed over the interval $(0, 80)$, so $\xi_T^{0.1} = 8.0$ and thus $\xi_Z^{0.9} = v^{8.0}$. ▼

The graph in Figure 4.2.1 can be used to establish relationships between the d.f. and p.d.f. of Z and those of T:

Relationship of Z to T for Whole Life Insurance



For $z \le 0$, $\{Z \le z\}$ is the null event

For $0 < z < 1$, $\{Z \le z\} = \{T \ge \log z / \log v\}$, and

For $z \ge 1$, $\{Z \le z\}$ is the certain event.

Therefore,

$$F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1 - F_T(\log z / \log v) & 0 < z < 1 \\ 1 & 1 \le z. \end{cases} \tag{4.2.7}$$

By differentiation of (4.2.7),

$$f_Z(z) = \begin{cases} f_T[(\log z)/(\log v)][1/(\delta z)] & 0 < z < 1 \\ 0 & \text{elsewhere.} \end{cases} \tag{4.2.8}$$

Example 4.2.2

For the assumptions in Example 4.2.1, determine

a. Z's d.f.

b. Z's p.d.f.

Solution:

a. From $F_T(t) = \begin{cases} t/80 & 0 \le t \le 80 \\ 1 & t \ge 80, \end{cases}$

we see that $\Pr\{T > 80\} = 0.0$, so $\Pr\{0 < Z < v^{80}\} = 0.0$. Therefore, from (4.2.7)

$$F_Z(z) = \begin{cases} 0 & z < v^{80} \\ 1 - [(\log z)/(\log v)]/80 & v^{80} < z < 1 \\ 1 & z \ge 1. \end{cases}$$

b. By differentiation of the d.f. in part (a),

$$f_Z(z) = \begin{cases} (1/80)(1/\delta z) & v^{80} < z < 1 \\ 0 & \text{elsewhere.} \end{cases}$$ ▼

We now turn our attention to a common application involving portfolios of risk: determining an initial investment fund for a segment of insurances in the total portfolio. The individual risk model and the normal approximation (as discussed in Section 2.4) are used.

Example 4.2.3

Assume that each of 100 independent lives

- Is age x
- Is subject to a constant force of mortality, $\mu = 0.04$, and
- Is insured for a death benefit amount of 10 units, payable at the moment of death.

The benefit payments are to be withdrawn from an investment fund earning $\delta = 0.06$. Calculate the minimum amount at $t = 0$ so that the probability is

approximately 0.95 that sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.

Solution:

For each life,

$$\begin{aligned} b_t &= 10 \qquad t \geq 0, \\ v_t &= v^t \qquad t \geq 0, \\ Z &= 10v^T \qquad T \geq 0. \end{aligned}$$

If we think of the lives as numbered, perhaps by the order of issuing policies, then at $t = 0$ the present value of all payments to be made is

$$S = \sum_{1}^{100} Z_j$$

where Z_j is the present value at $t = 0$ for the payment to be made at the death of the life numbered j.

We can use the fact that Z is 10 times the present-value random variable for the unit amount whole life insurance to calculate the mean and variance. For constant forces of interest, δ, and mortality, μ, the actuarial present value for the unit amount whole life insurance is

$$\bar{A}_x = \int_0^\infty e^{-\delta t} e^{-\mu t} \mu \, dt = \frac{\mu}{\mu + \delta}.$$

Then, for this example,

$$\mathrm{E}[Z] = 10\bar{A}_x = 10 \frac{0.04}{0.1} = 4,$$

$$\mathrm{E}[Z^2] = 10^2 {}^2\bar{A}_x = 100 \frac{0.04}{0.04 + 2(0.06)} = 25$$

and $\mathrm{Var}(Z) = 9$.

Using these values for the mean and the variance of each term in the sum for S, we have

$$\begin{aligned} \mathrm{E}[S] &= 100(4) = 400, \\ \mathrm{Var}(S) &= 100(9) = 900. \end{aligned}$$

Analytically, the required minimum amount is a number, h, such that

$$\Pr(S \leq h) = 0.95,$$

or equivalently

$$\Pr\left[\frac{S - \mathrm{E}[S]}{\sqrt{\mathrm{Var}(S)}} \leq \frac{h - 400}{30}\right] \doteq 0.95.$$

By use of a normal approximation, we obtain

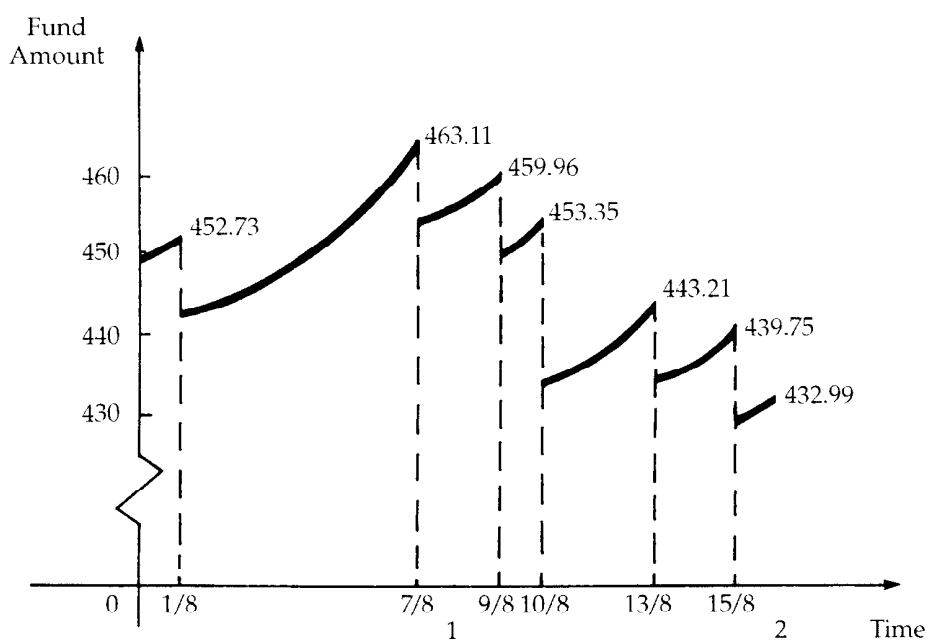
$$\frac{h - 400}{30} = 1.645,$$

$$h = 449.35. \qquad \blacktriangledown$$

Observations:

1. The 49.35 difference between this initial fund of 449.35 and the expectation of the present value of all payments, 400, is the risk loading of Chapter 1. The loading is 0.4935 per life, or 4.935% per unit payment, or 12.34% of the actuarial present value.
2. This example, like Examples 2.5.2 and 2.5.3, used the individual risk model and a normal approximation to the probability distribution of S. In the short-period examples, the collected income, equal to expected claims plus a risk loading, was determined to have a high probability of being in excess of claims. In this long-period life insurance example, the collected income plus interest income on it at the assumed interest rate is determined to be sufficient to cover the benefit payments. The initial fund of 449.35 will cover less than 45% of the eventual certain payout of 1,000.
3. A graph of the amount in the fund during the first 2 years for a payout pattern when one death occurs at each of times 1/8, 7/8, 9/8, 13/8, and 15/8, and two deaths occur at time 10/8 is shown in Figure 4.2.2. Between the benefit payments, represented by the discontinuities, are exponential arcs representing the growth of the fund at $\delta = 0.06$.

Graph of an Outcome for the Fund



4. There are infinitely many payout patterns, each with its own graph. Both the number of claims and the times of those claims affect the fund. For example, had the seven claims all occurred within the first instant, instead of the payout pattern of Figure 4.2.2, the fund would have dropped immediately to 379.35 and then grown to 427.72 by the end of the second year.

These examples illustrate the different roles of the three random elements in risk model building, that is, whether or not a claim will occur, the size, and the time of payment if one occurs. In Example 2.5.2 there was uncertainty about only the occurrence of the claim. In Example 4.2.2 there was uncertainty about only the time of claim payment. Other uncertainties have been ignored in these models. In Examples 4.2.1, 4.2.2, and 4.2.3 we have ignored the possibility of the fund earning interest at rates different from the deterministic rates assumed.

4.2.2 Endowment Insurance

An ***n-year pure endowment*** provides for a payment at the end of the n years if and only if the insured survives at least n years from the time of policy issue. If the amount payable is a unit, then

$$b_t = \begin{cases} 0 & t \le n \\ 1 & t > n, \end{cases}$$

$$v_t = v^n \qquad t \ge 0,$$

$$Z = \begin{cases} 0 & T \le n \\ v^n & T > n. \end{cases}$$

The only element of uncertainty in the pure endowment is whether or not a claim will occur. The size and time of payment, if a claim occurs, are predetermined. In the expression $Z = v^n Y$, Y is the indicator of the event of survival to age $x + n$. This Y has the value 1 if the insured survives to age $x + n$ and has the value 0 otherwise. The n-year pure endowment's actuarial present value has two symbols. In an insurance context it is $A_{x:\overline{n}|}^{\ 1}$. We see in the next chapter that it is denoted by ${}_nE_x$ in an annuity context. This distinction is not strict; the reader will have to be ready for either:

$$A_{x:\overline{n}|}^{\ 1} = \mathrm{E}[Z] = v^n \, \mathrm{E}[Y] = v^n \, {}_np_x,$$

and

$$\begin{aligned} \mathrm{Var}(Z) &= v^{2n} \, \mathrm{Var}(Y) = v^{2n} \, {}_np_x \, {}_nq_x \\ &= {}^2A_{x:\overline{n}|}^{\ 1} - (A_{x:\overline{n}|}^{\ 1})^2. \end{aligned} \tag{4.2.9}$$

An ***n-year endowment insurance*** provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first. If the insurance is for a unit amount and the death benefit is payable at the moment of death, then

$$b_t = 1 \qquad t \geq 0,$$

$$v_t = \begin{cases} v^t & t \leq n \\ v^n & t > n, \end{cases}$$

$$Z = \begin{cases} v^T & T \leq n \\ v^n & T > n. \end{cases}$$

The actuarial present value is denoted by $\bar{A}_{x:\overline{n}|}$. Since $b_t = 1$ for the endowment insurance, we have by the rule of moments

$$\mathrm{E}[Z^j] \,@\, \delta = \mathrm{E}[Z] \,@\, j\delta.$$

Moreover,

$$\mathrm{Var}(Z) = {}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2. \tag{4.2.10}$$

This insurance can be viewed as the combination of an n-year term insurance and an n-year pure endowment—each for a unit amount. Let Z_1, Z_2, and Z_3 denote the present-value random variables of the term, the pure endowment, and the endowment insurances, respectively, with death benefits payable at the moment of death of (x). From the preceding definitions we have

$$Z_1 = \begin{cases} v^T & T \leq n \\ 0 & T > n, \end{cases}$$

$$Z_2 = \begin{cases} 0 & T \leq n \\ v^n & T > n, \end{cases}$$

$$Z_3 = \begin{cases} v^T & T \leq n \\ v^n & T > n. \end{cases}$$

It follows that

$$Z_3 = Z_1 + Z_2, \tag{4.2.11}$$

and by taking expectations of both sides

$$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\,1}. \tag{4.2.12}$$

We can also find the $\mathrm{Var}(Z_3)$ by using (4.2.11),

$$\mathrm{Var}(Z_3) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) + 2\,\mathrm{Cov}(Z_1, Z_2). \tag{4.2.13}$$

By use of the formula

$$\mathrm{Cov}(Z_1, Z_2) = \mathrm{E}[Z_1 Z_2] - \mathrm{E}[Z_1]\,\mathrm{E}[Z_2] \tag{4.2.14}$$

and the observation that

$$Z_1 Z_2 = 0$$

for all T, we have

$$\mathrm{Cov}(Z_1, Z_2) = -\mathrm{E}[Z_1]\,\mathrm{E}[Z_2] = -\bar{A}^1_{x:\overline{n}|} A_{x:\overline{n}|}^{\,1}. \tag{4.2.15}$$

Substituting (4.2.5), (4.2.9), and (4.2.15) into (4.2.13) produces a formula for $\mathrm{Var}(Z_3)$ in terms of actuarial present values for an n-year term insurance and a pure endowment.

Since the actuarial present values are positive, the $\text{Cov}(Z_1, Z_2)$ is negative. This is to be anticipated since, of the pair Z_1 and Z_2, one is always zero and the other positive. On the other hand, the correlation coefficient of Z_1 and Z_2 is not -1 since they are not linear functions of each other; recall Exercise 1.23(c).

4.2.3 Deferred Insurance

An ***m-year deferred insurance*** provides for a benefit following the death of the insured only if the insured dies at least m years following policy issue. The benefit payable and the term of the insurance may be any of those discussed above. For example, an m-year deferred whole life insurance with a unit amount payable at the moment of death has

$$b_t = \begin{cases} 1 & t > m \\ 0 & t \le m, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T & T > m \\ 0 & T \le m. \end{cases}$$

The actuarial present value is denoted by ${}_{m|}\bar{A}_x$ and is equal to

$$\int_m^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt. \tag{4.2.16}$$

Example 4.2.4

Consider a 5-year deferred whole life insurance payable at the moment of the death of (x). The individual is subject to a constant force of mortality $\mu = 0.04$. For the distribution of the present value of the benefit payment, at $\delta = 0.10$:

a. Calculate the expectation
b. Calculate the variance
c. Display the distribution function
d. Calculate the median $\xi_Z^{0.5}$.

Solution:

a. For arbitrary forces μ and δ,

$${}_{5|}\bar{A}_x = \int_5^\infty e^{-\delta t} \, e^{-\mu t} \, \mu \, dt = \frac{\mu}{\mu + \delta} \, e^{-5(\mu+\delta)};$$

thus for $\mu = 0.04$ and $\delta = 0.10$,

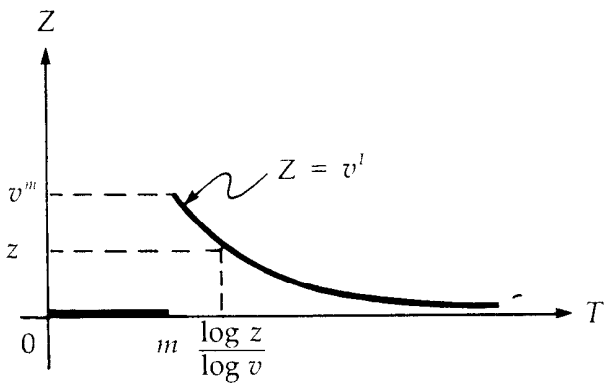
$${}_{5|}\bar{A}_x = \frac{2}{7} \, e^{-0.7} = 0.1419.$$

b. By the rule of moments,

$$\text{Var}(Z) = \frac{0.04}{0.04 + 0.20} \, e^{-5(0.04+0.20)} - \frac{4}{49} \, e^{-1.4} = 0.0301.$$

c. As for the case of whole life insurance, a graph of the relation between Z and T provides an outline for the solution. For the general m-year deferred whole life insurance, the graph is given in Figure 4.2.3.

Relationship of Z to T for Deferred Whole Life Insurance



Although T is a continuous random variable, Z is mixed with a probability mass at 0 because $Z = 0$ corresponds to $T \leq m$.

For general mortality assumptions and a constant force of interest, we have for $Z = 0$,

$$F_Z(0) = \Pr(T \leq m) = F_T(m); \tag{4.2.17}$$

for $0 < z < v^m$,

$$\begin{aligned} F_Z(z) &= \Pr(Z \leq z) = \Pr(Z = 0) + \Pr(0 < Z \leq z) \\ &= \Pr(T \leq m) + \Pr(0 < v^T \leq z) \\ &= \Pr(T \leq m) + \Pr\left(T > \frac{\log z}{\log v}\right) \\ &= F_T(m) + 1 - F_T\left(\frac{\log z}{\log v}\right); \end{aligned} \tag{4.2.18}$$

for $z > v^m$,

$$F_Z(z) = 1. \tag{4.2.19}$$

In this example of 5-year deferred whole life insurance where $\mu = 0.04$ and $\delta = 0.10$, we have

from (4.2.17),

$$F_Z(0) = F_T(5) = 1 - e^{-0.2} = 0.1813;$$

from (4.2.18) for $0 < z < v^5$,

$$F_Z(z) = F_T(5) + 1 - F_T\frac{\log z}{-0.1}$$

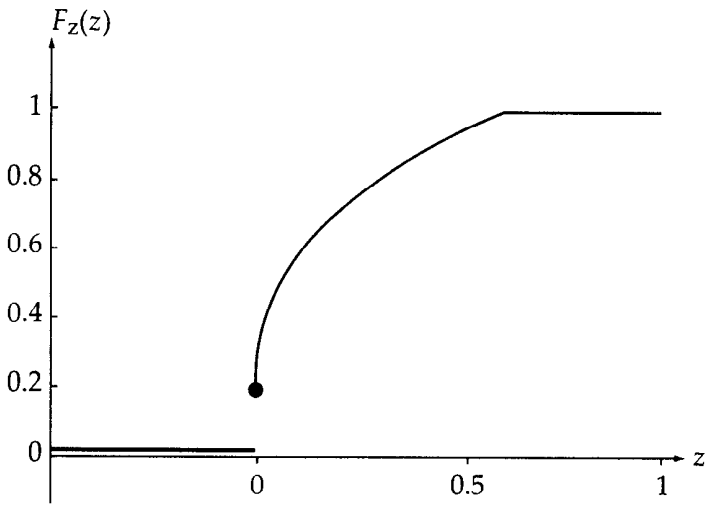
$$= 1 - e^{-0.2} + z^{0.04/0.10} = 0.1813 + z^{0.4}; \qquad (4.2.20)$$

from (4.2.19) for $z > v^5$,

$$F_Z(z) = 1.$$

The graph of this d.f. is shown in Figure 4.2.4.

Distribution Function of *Z*



d. From Figure 4.2.4 or (4.2.20), we see that the median is the solution of

$$0.5 = 0.1813 + z^{0.4}.$$

Thus, $\xi_Z^{0.5} = 0.0573$. ▼

Observations:

1. The largest value of Z with nonzero probability density in this example is $e^{-0.1(5)} = 0.6065$, corresponding to $T = 5$.
2. The distribution of Z in this example is highly skewed to the right. While its total mass is in the interval [0, 0.6065] and its mean is 0.1419, its median is only 0.0573. This skewness in the direction of large positive values is characteristic of many claim distributions in all fields of insurance.

4.2.4 Varying Benefit Insurance

The general model given by (4.2.1) can be used for analysis in most applications. We have used it with level benefit life insurances. It can also be applied to insurances where the level of the death benefit either increases or decreases in arithmetic

progression over all or a part of the term of the insurance. Such insurances are often sold as an additional benefit when a basic insurance provides for the return of periodic premiums at death or when an annuity contract contains a guarantee of sufficient payments to match its initial premium.

An ***annually increasing whole life insurance*** providing 1 at the moment of death during the first year, 2 at the moment of death in the second year, and so on, is characterized by the following functions:

$$\begin{aligned} b_t &= \lfloor t + 1 \rfloor & t \geq 0, \\ v_t &= v^t & t \geq 0, \\ Z &= \lfloor T + 1 \rfloor v^T & T \geq 0, \end{aligned}$$

where the $\lfloor\ \rfloor$ denote the greatest integer function.

The actuarial present value for such an insurance is

$$(I\bar{A})_x = \mathrm{E}[Z] = \int_0^\infty \lfloor t + 1 \rfloor\, v^t\, {}_tp_x\, \mu_x(t)\, dt.$$

The higher order moments are not equal to the actuarial present value at an adjusted force of interest as was the case for insurances with benefit payments equal to 0 or 1. These moments can be calculated directly from their definitions.

The increases in the benefit of the insurance can occur more, or less, frequently than once per year. For an m-thly increasing whole life insurance the benefit would be $1/m$ at the moment of death during the first m-th of a year of the term of the insurance, $2/m$ at the moment of death during the second m-th of a year during the term of the insurance, and so on, increasing by $1/m$ at m-thly intervals throughout the term of the insurance. For such a whole life insurance the functions are

$$\begin{aligned} b_t &= \frac{\lfloor tm + 1 \rfloor}{m} & t \geq 0, \\ v_t &= v^t & t \geq 0, \\ Z &= \frac{v^T \lfloor Tm + 1 \rfloor}{m} & T \geq 0. \end{aligned}$$

The actuarial present value is

$$(I^{(m)}\, \bar{A})_x = \mathrm{E}[Z].$$

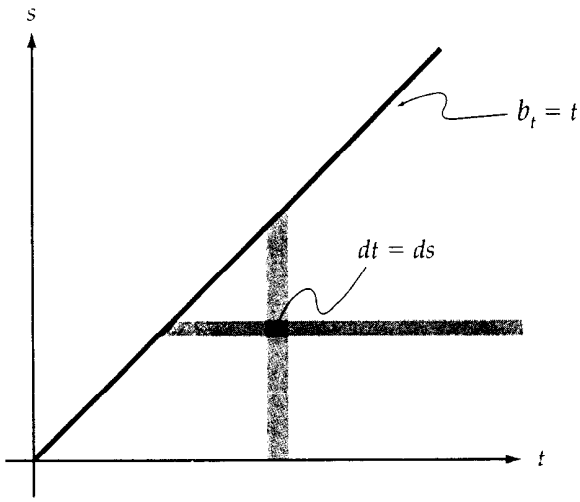
The limiting case, as $m \to \infty$ in the m-thly increasing whole life insurance, is an insurance paying t at the time of death, t. Its functions are

$$\begin{aligned} b_t &= t & t \geq 0, \\ v_t &= v^t & t \geq 0, \\ Z &= Tv^T & T \geq 0. \end{aligned}$$

Its actuarial present-value symbol is $(\bar{I}\bar{A})_x$.

This continuously increasing whole life insurance is equivalent to a set of deferred level whole life insurances. This equivalence is shown graphically in Figure 4.2.5 where the region between the line $b_t = t$ and the t-axis represents the insurance over the future lifetime. If the infinitesimal regions are joined in the vertical direction for a fixed t, the total benefit payable at t is obtained. If they are joined in the horizontal direction for a fixed s, an s-year deferred whole life insurance for the level amount ds is obtained.

Continuously Increasing Insurance



This equivalence implies that the actuarial present values for the coverages are equal. The equality can be established as follows.

By definition,

$$(\bar{I}\bar{A})_x = \int_0^\infty t\, v^t\, {}_tp_x\, \mu_x(t)\, dt,$$

and interpreting t in the integrand as the integral from zero to t in Figure 4.2.5 we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \left(\int_0^t ds \right) v^t\, {}_tp_x\, \mu_x(t)\, dt.$$

If we interchange the order of integration and, for each s value, integrate on t from s to x, we have

$$(\bar{I}\bar{A})_x = \int_0^\infty \int_s^\infty v^t\, {}_tp_x\, \mu_x(t)\, dt\, ds$$

$$= \int_0^\infty {}_{s|}\bar{A}_x\, ds$$

by (4.2.16).

If, for any of these m-thly increasing life insurances, the benefit is payable only if death occurs within a term of n years, the insurance is an **m-*thly increasing* n-*year term life insurance.***

Complementary to the anually increasing n-year term life insurance is the ***annually decreasing n-year term life insurance*** providing n at the moment of death during the first year, $n - 1$ at the moment of death during the second year, and so on, with coverage terminating at the end of the n-th year. Such an insurance has the following functions:

$$b_t = \begin{cases} n - \lfloor t \rfloor & t \le n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} v^T(n - \lfloor T \rfloor) & T \le n \\ 0 & T > n. \end{cases}$$

The actuarial present value for this insurance is

$$(D\bar{A})^1_{x:\overline{n}|} = \int_0^n v^t \, (n - \lfloor t \rfloor) \, {}_tp_x \, \mu_x(t) \, dt.$$

This insurance is complementary to the anually increasing n-year term insurance in the sense that the sum of their benefit functions is the constant $n + 1$ for the n-year term.

Table 4.2.1 is a summary of the models in this section. The insurance plan name appears in the first column followed by the benefit and discount functions that define it in terms of the future lifetime of the insured at policy issue. The present-value function, which is always derived as the product of the previous two functions, is shown next. In the fifth column the International Actuarial Notation for the actuarial present value is shown. In the last column, a reference is given to a footnote stating whether or not the rule of moments can be used in the calculation of higher order moments.

4.3 Insurances Payable at the End of the Year of Death

In the previous section we developed models for life insurances with death benefits payable at the moment of death. In practice, most benefits are considered payable at the moment of death and then earn interest until the payment is actually made. The models were built in terms of T, the future lifetime of the insured at policy issue. In most life insurance applications, the best information available on the probability distribution of T is in the form of a discrete life table. This is the probability distribution of K, the curtate-future-lifetime of the insured at policy issue, a function of T. In this and the following section we bridge this gap by building models for life insurances in which the size and time of payment of the death benefits depend only on the number of complete years lived by the insured from policy issue up to the time of death. We refer to these insurances simply as ***payable at the end of the year of death.***

Summary of Insurances Payable Immediately on Death

(1) Insurance Name	(2) Benefit Function b_t	(3) Discount Function v_t	(4) Present-Value Function z_t	(5) Actuarial Present Value	(6) Higher Moments
Whole life	1	v^t	v^t	$\bar{A}_x$	*
n-Year term	$1 \quad t \le n$ $0 \quad t > n$	v^t	$v^t \quad t \le n$ $0 \quad t > n$	$\bar{A}^1_{x:\overline{n}\rceil}$	*
n-Year pure endowment	$0 \quad t \le n$ $1 \quad t > n$	v^n	$0 \quad t \le n$ $v^n \quad t > n$	$A_{x:\overline{n}\rceil}^{\ 1}, {}_nE_x$	*
n-Year endowment	1	$v^t \quad t \le n$ $v^n \quad t > n$	$v^t \quad t \le n$ $v^n \quad t > n$	$\bar{A}_{x:\overline{n}\rceil}$	*
m-Year deferred n-Year term	$1 \quad m < t \le n + m$ $0 \quad t \le m, t > n + m$	v^t	$v^t \quad m < t \le n + m$ $0 \quad t \le m, t > n + m$	${}_{m\vert n}\bar{A}_x$	*
n-Year term increasing annually	$\lfloor t + 1 \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$\lfloor t + 1 \rfloor v^t \quad t \le n$ $0 \quad t > n$	$(I\bar{A})^1_{x:\overline{n}\rceil}$	†
n-Year term decreasing annually	$n - \lfloor t \rfloor \quad t \le n$ $0 \quad t > n$	v^t	$(n - \lfloor t \rfloor)v^t \quad t \le n$ $0 \quad t > n$	$(D\bar{A})^1_{x:\overline{n}\rceil}$	†
Whole life increasing m-thly	$\lfloor tm + 1 \rfloor / m$	v^t	$v^t \lfloor tm + 1 \rfloor / m$	$(I^{(m)}\bar{A})_x$	†

Note: b_t, v_t, and z_t are defined only for $t \ge 0$.

*The j-th moment is equal to the actuarial present value at j times the given force of interest, denoted by ${}^jA$ for $j > 1$. Then the variance is ${}^2A - A^2$, symbolically.

†Calculated directly from the definition, $E[Z^j]$.

Our model is in terms of functions of the curtate-future-lifetime of the insured. The benefit function, b_{k+1}, and the discount function, v_{k+1}, are, respectively, the benefit amount payable and the discount factor required for the period from the time of payment back to the time of policy issue when the insured's curtate-future-lifetime is k, that is, when the insured dies in year $k + 1$ of insurance. The present value, at policy issue, of this benefit payment, denoted by z_{k+1}, is

$$z_{k+1} = b_{k+1}v_{k+1}. \tag{4.3.1}$$

Measured from the time of policy issue, the insurance year of death is 1 plus the curtate-future-lifetime random variable, K, defined in Section 3.2.3. As in the previous section, we denote the present-value random variable z_{K+1}, by Z.

For an n-year term insurance providing a unit amount at the end of the year of death, we have

$$b_{k+1} = \begin{cases} 1 & k = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$v_{k+1} = v^{k+1},$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & \text{elsewhere.} \end{cases}$$

The actuarial present value for this insurance is given by

$$A^1_{x:\overline{n}|} = \mathrm{E}[Z] = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}. \tag{4.3.2}$$

Note that the International Actuarial Notation symbol for the actuarial present value of an insurance payable at the end of the year of death is the symbol for the corresponding insurance payable at the moment of death with the bar removed.

The rule of moments, with the appropriate changes in notation, also holds for insurances payable at the end of the year of death. For example, for the n-year term insurance above,

$$\mathrm{Var}(Z) = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2$$

where

$${}^2A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} e^{-2\delta(k+1)}\, {}_kp_x\, q_{x+k}.$$

In Section 3.5 recursion relations for life expectancies are derived and used to determine their values. Recursion relations for the term insurance actuarial present values can be derived algebraically from (4.3.2):

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k} = vq_x + \sum_{k=1}^{n-1} v^{k+1}\, {}_kp_x\, q_{x+k}$$

$$= vq_x + vp_x \sum_{k=1}^{n-1} v^k\, {}_{k-1}p_{x+1}\, q_{x+k}$$

$$= vq_x + vp_x \sum_{j=0}^{n-2} v^{j+1}\, {}_jp_{x+1}\, q_{x+1+j} = vq_x + vp_x\, A^1_{x+1:\overline{n-1}|}. \qquad (4.3.3)$$

For (4.3.3) to be true at $n = 1$, we define $A^1_{x:\overline{0}|} = 0.0$ for all x.

Note: On a select table basis, all x's in the subscripts in (4.3.3) would be enclosed in brackets.

Example 4.3.1

On the basis of the Illustrative Life Table and $i = 0.04$, determine the mean and variance of the present-value random variable for a 10-year term insurance with a unit benefit payable at the end of the year of death issued on (30).

Solution:

Starting with the initial value $A^1_{40:\overline{0}|} = 0.0$ and using (4.3.3) adapted to this insurance,

$$A^1_{30:\overline{10-k}|} = vq_{30+k} + vp_{30+k}\, A^1_{30+k+1:\overline{10-(k+1)}|} \quad k = 0, 1, \ldots, 8, 9,$$

we have by working from age 40 to age 30,

$$A^1_{30:\overline{10}|} = 0.01577285$$

and

$$\text{Var}(Z) = 0.01271978 - (0.01577285)^2 = 0.1247099.$$

These values were determined by the spreadsheet constructed in the Computing Exercises. ▼

For a whole life insurance issued to (x), the model may be obtained by letting $n \to \infty$ in the n-year term insurance model. For the actuarial present value we have

$$A_x = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x\, q_{x+k}. \qquad (4.3.4)$$

Multiplication of both sides of (4.3.4) by l_x yields

$$l_x A_x = \sum_{k=0}^{\infty} v^{k+1}\, d_{x+k}. \qquad (4.3.5)$$

Formula (4.3.5) shows the balance, at the time of policy issue, between the aggregate fund of actuarial present values for l_x lives insured at age x and the outflow of funds in accordance with the expected deaths of the l_x lives. It is a compound interest equation of value that is stated on an expected value basis.

The expression

$$\sum_{k=r}^{\infty} v^{k+1} d_{x+k} \tag{4.3.6}$$

is that part of the fund at issue that, together with interest at the assumed rate, will provide the payments for the expected deaths after the r-th insurance year.

Accumulation of (4.3.6) at the assumed interest rate for r years yields

$$\sum_{k=r}^{\infty} v^{k-r+1} d_{x+k}, \tag{4.3.7}$$

the expected amount in the fund after r insurance years. A comparison of expression (4.3.7) with (4.3.5) shows it to be $l_{x+r} A_{x+r}$. The difference between this amount and an actual fund is due to deviations of the actual deaths from the expected deaths (according to the life table adopted), and deviations of the actual interest income from the interest income at the assumed rate.

Example 4.3.2

A group of 100 lives age 30 set up a fund to pay 1,000 at the end of the year of death of each member to a designated survivor. Their mutual agreement is to pay into the fund an amount equal to the whole life insurance actuarial present value calculated on the basis of the Illustrative Life Table at 6% interest. The members, not selected by an insurance company, decided to use this population table as the basis of their plan. The actual experience of the fund is one death in each of the second and fifth years; interest income is at 6% in the first year, 6-1/2% in the second and third years, 7% in the fourth and fifth years. What is the difference, at the end of the first 5 years, between the expected size of the fund as determined at the inception of the plan and the actual fund?

Solution:

On the agreed bases, $1{,}000\ A_{30} = 102.4835$, so, for the 100 lives, the fund starts at 10,248.35. Also, $A_{35} = 0.1287194$ and $l_{35}/l_{30} = 0.9915040$.

For 100 lives age 30, the expected size of the fund after 5 years will be

$$(1{,}000)(100)\frac{l_{35}}{l_{30}} A_{35} = 12{,}762.58.$$

The development of the actual fund would be as follows, where F_k denotes its size at the end of insurance year k:

$$F_0 = 10{,}248.35$$

$$F_1 = (10{,}248.35)(1.06) = 10{,}863.25$$

$$F_2 = (10{,}863.25)(1.065) - 1{,}000 = 10{,}569.36$$

$$F_3 = (10{,}569.36)(1.065) = 11{,}256.37$$

$$F_4 = (11{,}256.37)(1.07) = 12{,}044.32$$

$$F_5 = (12{,}044.32)(1.07) - 1{,}000 = 11{,}887.42.$$

Thus the required difference is 12,762.58 − 11,887.42 = 875.16. This result combines the investment experience and the mortality experience for the 5-year period. There were gains from the investment earnings in excess of the assumed rate of 6%. On the other hand, there were losses on the mortality experience of two deaths as compared to the expected number of 0.8496. The interpretation of such results in terms of the various sources such as investment earnings and mortality is an actuarial responsibility. ▼

We derived the recursion relations for n-year term insurance actuarial present values (4.3.3) algebraically. Whereas the relationship will hold for whole life insurance actuarial present values as the limiting case of n-year term insurance, as n goes to ∞, we will establish the whole life insurance relationship independently to illustrate a probabilistic derivation.

Consider A_x from its definition $\mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1}]$. For emphasis we now write this as

$$A_x = \mathrm{E}[Z] = \mathrm{E}[v^{K(x)+1} | K(x) \geq 0],$$

which is redundant since all of $K(x)$'s probability is on the non-negative integers.

E[Z] can be calculated by considering the event that (x) dies in the first year, that is, $K(x) = 0$, and its complement, that (x) survives the first year, that is, $K(x) \geq 1$. We can write

$$\begin{aligned}\mathrm{E}[Z] = {} & \mathrm{E}[v^{K(x)+1} | K(x) = 0] \Pr[K(x) = 0] \\ & + \mathrm{E}[v^{K(x)+1} | K(x) \geq 1] \Pr[K(x) \geq 1].\end{aligned} \tag{4.3.8}$$

In this expression we can readily substitute

$$\mathrm{E}[v^{K(x)+1} | K(x) = 0] = v,$$

$$\Pr[K(x) = 0] = q_x,$$

and

$$\Pr[K(x) \geq 1] = p_x.$$

To find an expression for the remaining factor, we rewrite it as

$$\mathrm{E}[v^{K(x)+1} | K(x) \geq 1] = v\,\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0].$$

Since $K(x)$ is the curtate-future-lifetime of (x), given $K(x) \geq 1$, $K(x) - 1$ must be the curtate-future-lifetime of $(x + 1)$.

If we are willing to use the same probabilities for the conditional distribution of $K(x) - 1$ given $K(x) \geq 1$, as we would for a newly considered life age $x + 1$, then we may write

$$\mathrm{E}[v^{(K(x)-1)+1} | K(x) - 1 \geq 0] = A_{x+1} \tag{4.3.9}$$

and substitute it into (4.3.8) to obtain

$$A_x = vq_x + vA_{x+1}p_x. \tag{4.3.10}$$

The assumed equality,

(the distribution of the future lifetime
of a newly insured life aged $x + 1$)

= (the distribution of the future lifetime of a life now age $x + 1$ who was insured 1 year ago),

was discussed in Section 3.8. In terms of select tables, the right-hand side of (4.3.9) would be $A_{[x]+1}$. In (4.3.10) every x would be $[x]$.

Note that (4.3.10) is the same backward recursion formula as (4.3.3). That is,

$$u(x) = vq_x + vp_x\, u(x + 1).$$

It is the starting value that makes the solution the actuarial present value of whole life insurance or of n-year term insurance. We see this same recursion formula for the actuarial present values of n-year endowment insurance where the starting values are the endowment maturity value.

Analysis of relationship (4.3.10) can give more insight into the nature of A_x. After replacement of p_x by $1 - q_x$ and multiplication of both sides by $(1 + i)l_x$, (4.3.10) can be rearranged as

$$l_x\,(1 + i)A_x = l_x A_{x+1} + d_x(1 - A_{x+1}). \tag{4.3.11}$$

For the random survivorship group, this equation has the following interpretation. Together with 1 year's interest, A_x will provide A_{x+1} for all l_x lives and an additional $1 - A_{x+1}$ for those expected to die within the year. This latter amount for each expected death, that is, $q_x(1 - A_{x+1})$, is considered the ***annual cost of insurance.*** The A_{x+1} is set aside for survivors and deaths, the $1 - A_{x+1}$ is required only for a death.

Dividing by l_x and then subtracting $A_x + q_x(1 - A_{x+1})$ from both sides of (4.3.11), we have

$$A_{x+1} - A_x = iA_x - q_x(1 - A_{x+1}). \tag{4.3.12}$$

In words, the difference between the actuarial present values at age x and one later at age $x + 1$ is equal to the interest on the actuarial present value at x less the annual cost of insurance for the year.

Another expression for A_x can be obtained from (4.3.10) by replacing p_x by $1 - q_x$, multiplying both sides by v^x, and rearranging the terms to get

$$v^{x+1}A_{x+1} - v^xA_x = -v^{x+1}q_x(1 - A_{x+1}),$$

or

$$\Delta v^xA_x = -v^{x+1}q_x(1 - A_{x+1}).$$

Summing from $x = y$ to ∞ (see Appendix 5), we obtain

$$-v^yA_y = -\sum_{x=y}^{\infty} v^{x+1}q_x(1 - A_{x+1})$$

and thus

$$A_y = \sum_{x=y}^{\infty} v^{x+1-y}q_x(1 - A_{x+1}).$$

This expression shows that the actuarial present value at y is the present value at y of the annual costs of insurance over the remaining lifetime of the insured.

The n-year endowment insurance with a unit amount payable at the end of the year of death is a combination of the n-year term insurance of this section and the n-year pure endowment for a unit amount that was discussed in the previous section. Thus the functions for it are

$$b_{k+1} = 1 \qquad k = 0, 1, \ldots,$$

$$v_{k+1} = \begin{cases} v^{k+1} & k = 0, 1, \ldots, n-1 \\ v^n & k = n, n+1, \ldots, \end{cases}$$

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots. \end{cases}$$

The actuarial present value is

$$\bar{A}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x\,q_{x+k} + v^n\,{}_np_x. \tag{4.3.13}$$

The annually increasing whole life insurance, paying $k + 1$ units at the end of insurance year $k + 1$ provided the insured dies in that insurance year, has the benefit and discount functions and present-value random variable as follows:

$$b_{k+1} = k + 1 \qquad k = 0, 1, 2, \ldots,$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, 2, \ldots,$$

$$Z = (K + 1)\,v^{K+1} \qquad K = 0, 1, 2, \ldots.$$

The actuarial present value is denoted by $(IA)_x$.

The annually decreasing n-year term insurance, during the n-year period, provides a benefit at the end of the year of death in an amount equal to $n - k$ where

k is the number of complete years lived by the insured since issue. Its functions are

$$b_{k+1} = \begin{cases} n - k & k = 0, 1, \ldots, n-1 \\ 0 & k = n, n+1, \ldots, \end{cases}$$

$$v_{k+1} = v^{k+1} \qquad k = 0, 1, \ldots,$$

$$Z = \begin{cases} (n - K)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots. \end{cases}$$

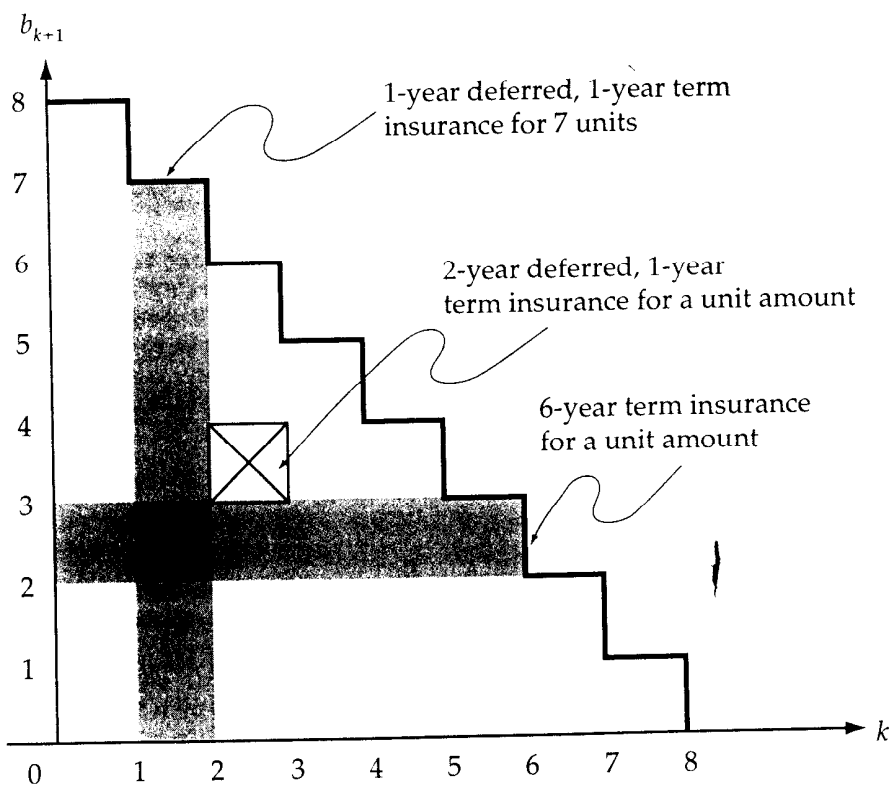
The actuarial present-value symbol for this insurance is $(DA)^1_{x:\overline{n}|}$.

As illustrated by Figure 4.2.5 for insurances payable at the moment of death, annually increasing insurances payable at the end of the year of death are equivalent to a combination of deferred level insurances each for a unit amount. Similarly, annually decreasing term insurances are equivalent to a combination of level term insurances of various term lengths. Figure 4.3.1 illustrates this for an annually decreasing 8-year term insurance.

Figure 4.3.1 shows the graph of the benefit function b_{k+1}. Each unit square region between the horizontal steps and the k-axis represents a deferred 1-year term insurance. When these are summed vertically, the deferred 1-year term insurances

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Annually Decreasing 8-Year Term Insurance



for the decreasing amounts are obtained. When the squares are summed horizontally, the level amount term insurances of varying duration are obtained. These vertical and horizontal sums are also indicated in Figure 4.3.1.

The equality of the actuarial present values for the combination of level term insurances and the combination of deferred term insurances can be demonstrated analytically. Thus, by definition

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} (n-k)\, v^{k+1}\, {}_kp_x\, q_{x+k}$$

$$= \sum_{k=0}^{n-1} (n-k)\, (v^k\, {}_kp_x)\, (vq_{x+k})$$

$$= \sum_{k=0}^{n-1} (n-k)\, {}_{k|1}A_x, \qquad (4.3.14)$$

the total of the column sums.

In (4.3.14) we can substitute

$$n-k = \sum_{j=0}^{n-k-1} (1)$$

to obtain

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k}.$$

By interchanging the order of summation we obtain

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} (1)\, v^{k+1}\, {}_kp_x\, q_{x+k},$$

and then by comparing the inner summation to (4.3.2), we can write

$$(DA)^1_{x:\overline{n}|} = \sum_{j=0}^{n-1} A^1_{x:\overline{n-j}|}.$$

Table 4.3.1 provides a summary of functions and symbols for the elementary insurances payable at the end of the year of death.

We close this section with a summary of the recursion relations for the actuarial present values of the insurances payable at the end of the year of death. Consider the list on page 119 arranged in the order of the entries of Table 4.3.1. Each entry is arranged across its line as the recursion relation, the domain for the relation, and the initial condition. Values for the actuarial present value would be generated from the lowest age of the mortality table to age y or ω.

Summary of Insurances Payable at End of Year of Death

(1) Insurance Name	(2) Benefit Function b_{k+1}	(3) Discount Function v_{k+1}	(4) Present-Value Function z_{k+1}	(5) Actuarial Present Value	(6) Higher Moments
(a) Whole life	1	v^{k+1}	v^{k+1}	A_x	*
(b) n-Year term	$1 \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$A^1_{x\overline{n}\rceil}$	*
(c) n-Year endowment	1	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $v^n \quad k = n, n+1, \ldots$	$v^{k+1} \quad k = 0, 1, \ldots, n-1$ $v^n \quad k = n, n+1, \ldots$	$A_{x\overline{n}\rceil}$	*
(d) m-Year deferred n-year term	$1 \quad k = m, m+1, \ldots, m+n-1$ $0 \quad k = 0, \ldots, m-1$ $\quad k = m+n, \ldots$	v^{k+1}	$v^{k+1} \quad k = m, m+1, \ldots, m+n-1$ $0 \quad k = 0, \ldots, m-1$ $\quad k = m+n, \ldots$	${}_{m\vert n}A_x$	*
(e) n-Year term increasing annually	$k+1 \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$(k+1)v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$(IA)^1_{x\overline{n}\rceil}$	†
(f) n-Year term decreasing annually	$n-k \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	v^{k+1}	$(n-k)v^{k+1} \quad k = 0, 1, \ldots, n-1$ $0 \quad k = n, n+1, \ldots$	$(DA)^1_{x\overline{n}\rceil}$	†
(g) Whole life increasing annually	$k+1 \quad k = 0, 1, \ldots$	v^{k+1}	$(k+1)v^{k+1} \quad k = 0, 1, \ldots$	$(IA)_x$	†

b_{k+1}, v_{k+1}, and z_{k+1} are defined only for non-negative integral values of k.
*Rule of moments holds, thus $\mathrm{Var}(Z) = {}^2A - A^2$ symbolically.
†Rule of moments does not hold.

(a) $A_x = vq_x + vp_x A_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$

and $A_\omega = 0.$

(b) $A^1_{x:\overline{y-x}|} = vq_x + vp_x A^1_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$

and $A^1_{y:\overline{0}|} = 0.$

(c) $A_{x:\overline{y-x}|} = vq_x + vp_x A_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y - 1,$

and $A_{y:\overline{0}|} = 1.$

(d) ${}_{y-x}|_n A_x = 0 + vp_x \, {}_{y-(x+1)}|_n A_{x+1} \qquad x = 0, 1, \ldots, y - 1,$

and ${}_0|_n A_y = A^1_{y:\overline{n}|}.$

(e) $(IA)^1_{x:\overline{y-x}|} = [vq_x + vp_x A^1_{x+1:\overline{y-(x+1)}|}] + vp_x (IA)^1_{x+1:\overline{y-(x+1)}|}$

$x = 0, 1, \ldots, y - 1,$ and $(IA)^1_{y:\overline{0}|} = 0.$

(f) $(DA)^1_{x:\overline{y-x}|} = (y - x)vq_x + vp_x (DA)^1_{x+1:\overline{y-(x+1)}|}$

$x = 0, 1, \ldots, y - 1,$ and $(DA)^1_{y:\overline{0}|} = 0.$

(g) $(IA)_x = [vq_x + vp_x A_{x+1}] + vp_x (IA)_{x+1} \qquad x = 0, 1, \ldots, \omega - 1,$

and $(IA)_\omega = 0.$

Observations:

1. Only (a) and (b) have been justified in this section. Arguments for (c) through (g) are similar to those for (a) and (b).
2. All seven equations are of the form

$$u(x) = c(x) + vp_x \, u(x + 1),$$

where $c(x)$ is a given function defined for the domain of the relation. In the language of difference equations, all seven equations have the same corresponding homogeneous equation, $u(x) = vp_x \, u(x + 1)$. It is linear but does not have constant coefficients.

3. Since $c(x) = vq_x$ for (a), (b), and (c), those actuarial present values are all solutions of the same recursion formula and are distinguished only by their starting values.

4.4 Relationships between Insurances Payable at the Moment of Death and the End of the Year of Death

We begin the study of these relationships with an analysis of the actuarial present value for whole life insurance paying a unit benefit at the moment of death. From (4.2.6) we have

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + \int_1^\infty v^t \, {}_tp_x \, \mu_x(t) \, dt.$$

The change of variables $s = t - 1$ in the second integral gives

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + v \int_0^\infty v^s \, {}_{s+1}p_x \, \mu_x(s + 1) \, ds. \tag{4.4.1}$$

On an aggregate mortality basis

$${}_{s+1}p_x \, \mu_x(s + 1) = p_x \, {}_sp_{x+1} \, \mu(x + s + 1)$$

so the second term of (4.4.1) would be $vp_x\bar{A}_{x+1}$. On a select mortality basis the second term would be $vp_{[x]}\bar{A}_{[x]+1}$. Returning to (4.4.1) and using aggregate notation, we have

$$\bar{A}_x = \int_0^1 v^t \, {}_tp_x \, \mu_x(t) \, dt + vp_x \, \bar{A}_{x+1} = \bar{A}^1_{x:\overline{1}|} + vp_x \, \bar{A}_{x+1}. \tag{4.4.2}$$

The integral in (4.4.2) can be expressed in discrete life table functions by adopting one of the assumptions about the form of the mortality function between integers as discussed in Section 3.6.

Under the assumption of a uniform distribution of deaths over each year of age,

$${}_tp_y \, \mu_y(t) = q_y \qquad 0 \le t \le 1, \text{ and } y = 0, 1, \ldots$$

which can be placed in (4.4.2) to obtain

$$\bar{A}_x = q_x \int_0^1 v^t \, dt + vp_x\bar{A}_{x+1},$$

$$= \frac{i}{\delta} vq_x + vp_x \, \bar{A}_{x+1}. \tag{4.4.3}$$

The domain for this relationship is $x = 0, 1, \ldots, \omega - 1$, and the starting value is $\bar{A}_\omega = 0$.

If we multiply both sides of recursion formula (a) by i/δ, we have

$$\frac{i}{\delta} A_x = \frac{i}{\delta} vq_x + vp_x \left(\frac{i}{\delta} A_{x+1} \right).$$

Since (a) and (4.4.3) embody the same recursion formula and have the same domain and the same initial value of 0 at ω, $(i/\delta)A_x$ is the solution for (4.4.3), and

$$\bar{A}_x = \frac{i}{\delta} A_x. \tag{4.4.4}$$

Formula (4.4.4) might have been anticipated under the assumption of a uniform distribution of deaths between integral ages. The effect of the assumption is to make the unit payable at the moment of death equivalent to a unit payable continuously throughout the year of death. With respect to interest, a unit payable continuously over the year is equivalent to i/δ at the end of the year.

The identity in (4.4.4) can be reached using the properties of the future-lifetime random variable under the assumption of a uniform distribution of deaths in each year of age as developed in Section 3.6. From (3.6.1) we write $T = K + S$. We observed there that, under the assumption of a uniform distribution of deaths in each year of age, K and S are independent and S has a uniform distribution over the unit interval. As corollaries to these observations, $K + 1$ and $1 - S$ are also independent, and $1 - S$ has a uniform distribution over the unit interval. In the identity

$$\bar{A}_x = \mathrm{E}[v^T] = \mathrm{E}[v^{K+1}(1 + i)^{1-S}],$$

we can use the independence of $K + 1$ and $1 - S$ to calculate the expectation of the product as the product of the expectations,

$$\mathrm{E}[v^{K+1}(1 + i)^{1-S}] = \mathrm{E}[v^{K+1}]\,\mathrm{E}[(1 + i)^{1-S}]. \tag{4.4.5}$$

The first factor on the right-hand side is A_x. Since $1 - S$ has the uniform distribution over the unit interval, the second factor is

$$\mathrm{E}[(1 + i)^{1-S}] = \int_0^1 (1 + i)^t\, 1\, dt = \frac{i}{\delta}.$$

Hence, again we have $\bar{A}_x = (i/\delta)A_x$ under the assumption of uniform distribution of deaths in each year of age.

A similar argument, again based on the assumption of a uniform distribution of deaths in each year of age, can be used to show that the actuarial present value of a whole life insurance which pays a unit at the end of the m-th of a year of death is equal to

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x. \tag{4.4.6}$$

This argument is outlined in Exercise 4.19.

In Section 3.6 we also discussed the assumption that the force of mortality is constant between integral ages. The relationship between the actuarial present values for whole life insurances payable at the moment of death and at the end of the year of death under this assumption is developed in Exercise 4.19. Since the hyperbolic assumption implies that the force of mortality decreases over the year of age (see Exercise 3.27), it is seldom realistic for human lives. Moreover, it leads to more complicated relationships that we will not develop here.

Next we turn to an analysis of the annually increasing n-year term insurance payable at the moment of death. For this insurance, the present-value random variable is

$$Z = \begin{cases} \lfloor T + 1 \rfloor v^T & T < n \\ 0 & T \geq n. \end{cases}$$

Since $\lfloor T + 1 \rfloor = K + 1$, we can use the relation $T = K + S$ to obtain

$$Z = \begin{cases} (K+1)v^{K+1}v^{S-1} & T < n \\ 0 & T \geq n. \end{cases}$$

If we let W be the present-value random variable for the annually increasing n-year term insurance payable at the end of the year of death,

$$W = \begin{cases} (K+1)v^{K+1} & K = 0, 1, \ldots, n-1 \\ 0 & K = n, n+1, \ldots. \end{cases}$$

Then

$$Z = W(1+i)^{1-S}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[W(1+i)^{1-S}].$$

Since W is a function of $K+1$ alone and $K+1$ and $1-S$ are independent,

$$\mathrm{E}[Z] = \mathrm{E}[W]\ \mathrm{E}[(1+i)^{1-S}]$$

$$= (IA)^1_{x:\overline{n}|}\frac{i}{\delta}.$$

These results for the whole life and the increasing term insurances payable at the moment of death, under the assumption of a uniform distribution of deaths over each year of age, are very similar,

$$\bar{A}_x = \frac{i}{\delta}A_x$$

and

$$(I\bar{A})^1_{x:\overline{n}|} = \frac{i}{\delta}(IA)^1_{x:\overline{n}|}.$$

Let us look at the general model to find the basis of the similarities. From (4.2.2),

$$Z = b_T v_T. \tag{4.4.7}$$

For the two continuous insurances above, the conditions used were

- $v_T = v^T$, and
- b_T was a function of only the integral part of T, the curtate-future-lifetime, K.

Writing this latter property as $b_T = b^*_{K+1}$ we can write (4.4.7) for these insurances as

$$Z = b^*_{K+1}v^T$$

$$= b^*_{K+1}v^{K+1}(1+i)^{1-S}$$

and

$$\mathrm{E}[Z] = \mathrm{E}[b^*_{K+1}v^{K+1}(1+i)^{1-S}]. \tag{4.4.8}$$

Under the assumption of a uniform distribution of deaths over each year of age, we can infer the independence of K and S and that $1-S$ also has a uniform distribution. Then we can write (4.4.8) as

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}[b^*_{K+1} v^{K+1}]\ \mathrm{E}[(1+i)^{1-S}] \\ &= \mathrm{E}[b^*_{K+1} v^{K+1}]\ \frac{i}{\delta}. \end{aligned} \tag{4.4.9}$$

Example 4.4.1

Calculate the actuarial present value and the variance for a 10,000 benefit, 30-year endowment insurance providing the death benefit at the moment of death of a male age 35 at issue of the policy. Use the Illustrative Life Table, the uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

For endowment insurance, $v_T \neq v^T$. Therefore, we cannot apply (4.4.9) directly. Recalling (4.2.11), which showed the endowment insurance as the sum of a term insurance and pure endowment, we can apply (4.4.9) to the term insurance component and then calculate the pure endowment insurance part. Thus, using (4.2.12) and (4.2.10), we can calculate the actuarial present value as follows:

$$\begin{aligned} \bar{A}_{35:\overline{30}|} &= \frac{i}{\delta} A^1_{35:\overline{30}|} + A_{35:\overline{30}|}^{\ 1} \\ &= (1.0297087)[0.06748179] + 0.1392408 \\ &= 0.208727, \end{aligned}$$

and the variance as

$$\begin{aligned} \mathrm{Var}(Z) &= {}^2\bar{A}_{35:\overline{30}|} - (\bar{A}_{35:\overline{30}|})^2 \\ &= 0.0309294 + 0.0242432 - (0.208727)^2 \\ &= 0.011606. \end{aligned}$$

For the 10,000 sum insured, $10{,}000\ \bar{A}_{35:\overline{30}|} = 2{,}087.27$ and $(10{,}000)^2\ \mathrm{Var}(Z) = 1{,}160{,}600$. ▼

Example 4.4.2

Calculate, for a life age 50, the actuarial present value for an annually decreasing 5-year term insurance paying 5,000 at the moment of death in the first year, 4,000 in the second year, and so on. Use the Illustrative Life Table, uniform distribution of deaths over each year of age assumption, and $i = 0.06$.

Solution:

Referring to Table 4.2.1, we see that

$$b_t = \begin{cases} 5 - \lfloor t \rfloor & t \leq 5 \\ 0 & t > 5 \end{cases}$$

is a function of only k, the integral part of t, and hence we may write it as

$$b_t = \begin{cases} 5 - k & k = 0, 1, 2, 3, 4 \\ 0 & k > 4. \end{cases}$$

The discount function is v^t, so we have

$$\begin{aligned}(D\bar{A})^1_{50:\overline{5}|} &= \frac{i}{\delta}(DA)^1_{50:\overline{5}|} \\ &= (1.0297087)\sum_{k=0}^{4}(5-k)\,v^{k+1}\,{}_kp_{50}\,q_{50+k} \\ &= (1.0297087)\frac{\sum_{k=0}^{4}(5-k)\,v^{k+1}\,d_{50+k}}{l_{50}} \\ &= 0.088307.\end{aligned}$$

Then, $1{,}000\,(D\bar{A})^1_{50:\overline{5}|} = 88.307$. ▼

For an insurance providing a death benefit at the moment of death that is not a function of K, further analysis is required to express its values in terms of those for an insurance payable at the end of the year of death. Consider the continuously increasing whole life insurance payable at the moment of death. This insurance is discussed extensively in Section 4.2, and its benefit function analyzed in Figure 4.2.5. Its functions are

$$\begin{aligned} b_t &= t \qquad t > 0, \\ v_t &= v^t \qquad t > 0, \\ z_t &= tv^t \qquad t > 0. \end{aligned}$$

To find $(\bar{I}\bar{A})_x$ we rewrite

$$\begin{aligned} Z &= (K+S)v^{K+S} \\ &= (K+1)v^{K+S} - (1-S)v^{K+1}(1+i)^{1-S} \\ &= (K+1)v^{K+1}(1+i)^{1-S} - v^{K+1}(1-S)(1+i)^{1-S}. \end{aligned}$$

Now taking expectations, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}[(K+1)v^{K+1}]\,\mathrm{E}[(1+i)^{1-S}] - \mathrm{E}[v^{K+1}]\,\mathrm{E}[(1-S)(1+i)^{1-S}] \\ &= (IA)_x\frac{i}{\delta} - A_x\,\mathrm{E}[(1-S)(1+i)^{1-S}]. \end{aligned}$$

We can simplify the last factor directly since $1 - S$ has a uniform distribution,

$$\mathrm{E}[(1-S)(1+i)^{1-S}] = \int_0^1 u(1+i)^u\,du = (\bar{D}\bar{s})_{\overline{1}|} = \frac{1+i}{\delta} - \frac{i}{\delta^2}.$$

Thus, we can write

$$(\bar{I}\bar{A})_x = \frac{i}{\delta}\left[(IA)_x - \left(\frac{1}{d} - \frac{1}{\delta}\right)A_x\right].$$

4.5 Differential Equations for Insurances Payable at the Moment of Death

Recursive-type expressions can be established for insurances payable at the moment of death. These are developed using calculus and lead to differential equations.

For a whole life insurance on (x),

$$\frac{d}{dx}\bar{A}_x = -\mu(x) + \bar{A}_x[\delta + \mu(x)] = \delta\bar{A}_x - \mu(x)(1 - \bar{A}_x), \tag{4.5.1}$$

which are the continuous analogues of (4.3.12). The notation used here is for an aggregate mortality basis. Verification of these expressions has been left to Exercise 4.21.

On the other hand, (4.5.1) can be developed from the definition of $\bar{A}_x$ by using conditional expectation as we did for A_x:

$$\begin{aligned}\bar{A}_x &= \mathrm{E}[v^T] \\ &= \mathrm{E}[v^T|T \le h]\ \Pr(T \le h) + \mathrm{E}[v^T|T > h]\ \Pr(T > h).\end{aligned} \tag{4.5.2}$$

Now,

$$\Pr(T \le h) = {}_hq_x \quad \text{and} \quad \Pr(T > h) = {}_hp_x, \tag{4.5.3}$$

and the conditional p.d.f. of T given $T \le h$ is

$$f_T(t|T \le h) = \begin{cases} \dfrac{f_T(t)}{F_T(h)} = \dfrac{{}_tp_x\,\mu(x+t)}{{}_hq_x} & 0 \le t \le h \\ 0 & \text{elsewhere.} \end{cases}$$

Thus,

$$\mathrm{E}[v^T|T \le h] = \int_0^h v^t\,\frac{{}_tp_x\,\mu(x+t)}{{}_hq_x}\,dt. \tag{4.5.4}$$

As we did in the expression for A_x, we will write

$$\begin{aligned}\mathrm{E}[v^T|T > h] &= v^h\mathrm{E}[v^{T-h}|(T-h) > 0] \\ &= v^h\bar{A}_{x+h}.\end{aligned} \tag{4.5.5}$$

Substitution of (4.5.3), (4.5.4), and (4.5.5) into (4.5.2) yields

$$\bar{A}_x = \int_0^h v^t\,\frac{{}_tp_x\,\mu(x+t)}{{}_hq_x}\,dt\ {}_hq_x + v^h\,\bar{A}_{x+h}\,{}_hp_x. \tag{4.5.6}$$

Then, on both sides of (4.5.6), we multiply by -1, add $\bar{A}_{x+h}$, and divide by h to obtain

$$\frac{\bar{A}_{x+h} - \bar{A}_x}{h} = \frac{-1}{h}\int_0^h v^t\, {}_tp_x\, \mu(x+t)\, dt + \bar{A}_{x+h}\frac{1 - v^h\, {}_hp_x}{h}. \tag{4.5.7}$$

Next,

$$\lim_{h\to 0}\frac{1}{h}\int_0^h v^t\, {}_tp_x\, \mu(x+t)\, dt = \frac{d}{ds}\int_0^s v^t\, {}_tp_x\, \mu(x+t)\, dt\Big|_{s=0} = \mu(x)$$

and

$$\lim_{h\to 0}\frac{1 - v^h\, {}_hp_x}{h} = -\frac{d}{dt}\,(v^t\, {}_tp_x)\Big|_{t=0} = \mu(x) + \delta.$$

Using these two limits as $h \to 0$ in (4.5.7) we obtain (4.5.1),

$$\frac{d}{dx}\bar{A}_x = -\mu(x) + \bar{A}_x[\mu(x) + \delta].$$

Solutions of this differential equation are outlined in Exercise 4.22.

4.6 Notes and References

The life contingencies textbooks listed in Appendix 6 give other developments of formulas for life insurance actuarial present values. Commutation functions, which were fundamental to actuarial calculations until this quarter century, are employed extensively in Jordan (1967).

There is little material in these textbooks on the concept of the time-until-death of an insured as a random variable. Until recently, research and exposition on this concept has been called ***individual risk theory.*** Cramér (1930) gives a detailed exposition of the ideas up to that time. Kahn (1962) and Seal (1969) give concise bibliographical information on both research and expository papers over a 100-year span.

Since 1970 there has been interest in actuarial models that consider both the time-until-death and the investment-rate-of-return as random variables. This combination is discussed in Chapter 21.

Exercises

Assume, unless otherwise stated, that insurances are payable at the moment of death, and that the force of interest is a constant δ with i and d as the equivalent rates of interest and discount.

Section 4.2

4.1. If $\mu(x) = \mu$, a positive constant, for all $x > 0$, show that $\bar{A}_x = \mu/(\mu + \delta)$.

4.2. Let $\mu(x) = 1/(1 + x)$, for all $x > 0$.

a. Integrate by parts to show that

$$\bar{A}_x = 1 - \delta \int_0^\infty e^{-\delta t} \frac{1 + x}{1 + x + t}\, dt.$$

b. Use the expression in (a) to show that $d\bar{A}_x/dx < 0$ for all $x > 0$.

4.3. Show that $d\bar{A}_x/di = -v(\bar{I}\bar{A})_x$.

4.4. Show that the expressions for the variance of the present value of an n-year endowment insurance paying a unit benefit, as given by (4.2.10) and (4.2.13), are identical.

4.5. Let Z_1 and Z_2 be as defined for equation (4.2.11).
a. Show that $\lim_{n\to 0} \text{Cov}(Z_1, Z_2) = \lim_{n\to\infty} \text{Cov}(Z_1, Z_2) = 0$.
b. Develop an implicit equation for the term of the endowment for which $\text{Cov}(Z_1, Z_2)$ is minimized.
c. Develop a formula for the minimum in (b).
d. Simplify the formulas in (b) and (c) for the case when the force of mortality is a constant μ.

4.6. Assume mortality is described by $l_x = 100 - x$ for $0 \le x \le 100$ and that the force of interest is $\delta = 0.05$.
a. Calculate $\bar{A}^1_{40:\overline{25}|}$.
b. Determine the actuarial present value for a 25-year term insurance with benefit amount for death at time t, $b_t = e^{0.05t}$, for a person age 40 at policy issue.

4.7. Assuming De Moivre's survival function with $\omega = 100$ and $i = 0.10$, calculate
a. $\bar{A}^1_{30:\overline{10}|}$
b. The variance of the present value, at policy issue, of the benefit of the insurance in (a).

4.8. If $\delta_t = 0.2/(1 + 0.05t)$ and $l_x = 100 - x$ for $0 \le x \le 100$, calculate
a. For a whole life insurance issued at age x, the actuarial present value and the variance of the present value of the benefits
b. $(\bar{I}\bar{A})_x$.

4.9. a. Show that $\bar{A}_x$ is the moment generating function of T, the future lifetime of (x), evaluated at $-\delta$.
b. Show that if T has a gamma distribution with parameters α and β, then $\bar{A}_x = (1 + \delta/\beta)^{-\alpha}$.

4.10. Given $b_t = t$, $\mu_x(t) = \mu$, and $\delta_t = \delta$ for all $t > 0$, derive expressions for
a. $(\bar{I}\bar{A})_x = \text{E}[b_T v^T]$ b. $\text{Var}(b_T v^T)$.

4.11. The random variable Z is the present-value random variable for a whole life insurance of unit amount payable at the moment of death and issued to (x). If $\delta = 0.05$ and $\mu_x(t) = 0.01$:
   a. Display the formula for the p.d.f. of Z.
   b. Graph the p.d.f. of Z.
   c. Calculate $\bar{A}_x = \mathrm{E}[Z]$ and $\mathrm{Var}(Z)$.

4.12. The random variable Z is the present-value random variable for an n-year endowment insurance as defined in Section 4.2.2. Exhibit the d.f. of Z in terms of the d.f. of T.

4.13. The random variable Z is defined as in Exercise 4.12. If $\delta = 0.05$, $\mu_x(t) = 0.01$, and $n = 20$:
   a. Display the d.f. of Z.
   b. Graph the d.f. of Z.
   c. Calculate $\bar{A}_{x:\overline{n}|} = \mathrm{E}[Z]$ using the distribution of Z. [Hint: Consider using the complement of the d.f.]

Section 4.3

4.14. If $l_x = 100 - x$ for $0 \le x \le 100$ and $i = 0.05$, evaluate
   a. $A_{40:\overline{25}|}$ b. $(IA)_{40}$.

4.15. Show that $A_{x:\overline{n}|} = A^1_{x:\overline{m}|} + v^m \, {}_mp_x \, A_{x+m:\overline{n-m}|}$ for $m < n$ and interpret the result in words.

4.16. If $A_x = 0.25$, $A_{x+20} = 0.40$, and $A_{x:\overline{20}|} = 0.55$, calculate
   a. $A^{\frac{1}{}}_{x:\overline{20}|}$ b. $A^1_{x:\overline{20}|}$.

4.17. a. Describe the benefits of the insurance with actuarial present value given by the symbol $(IA)_{x:\overline{m}|}$.
   b. Express the actuarial present value of (a) in terms of the symbols given in Tables 4.2.1 and 4.3.1.

4.18. In Example 4.3.2, let the expected size of the fund k years after the agreement, and immediately after the payment of death claims, be denoted by E_k, where $E_0 = (100)(1{,}000\ A_{30}) = 10{,}248.35$.
   a. Start with (4.3.10) and develop the forward recursion formula
$$E_k = 1.06E_{k-1} - 100{,}000_{k-1|}q_{30}.$$
   b. Use the recursion formula developed in (a) to confirm that $E_5 = 12{,}762.58$.

Section 4.4

4.19. Consider the timescale measured in intervals of length $1/m$ where the unit is a year. Let a whole life insurance for a unit amount be payable at the end of the m-thly interval in which death occurs. Let k be the number of complete insurance years lived prior to death and let j be the number of complete m-ths of a year lived in the year of death.

a. What is the present-value function for this insurance?
b. Set up a formula analogous to (4.4.2) for the actuarial present value, $A_x^{(m)}$, for this insurance.
c. Show algebraically that, under the assumption of a uniform distribution of deaths over the insurance year of age,

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

4.20. Show, under the assumption of a constant force of mortality between integral ages, that

$$\bar{A}_x = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x \, \mu_x(k) \frac{i + q_{x+k}}{\delta + \mu_x(k)}$$

where $\mu_x(k) = -\log p_{x+k}$.

Section 4.5

4.21. a. Show that (4.2.6), for an aggregate mortality basis, can be rewritten as

$$\bar{A}_x = \frac{1}{{}_xp_0 \, v^x} \int_x^{\infty} v^y \, {}_yp_0 \, \mu(y) \, dy \qquad x \geq 0.$$

b. Differentiate the formula of (a) to establish (4.5.1),

$$\frac{d\bar{A}_x}{dx} = [\mu(x) + \delta]\bar{A}_x - \mu(x) \qquad x \geq 0.$$

c. Use the same technique to show

$$\frac{d\bar{A}^1_{x:\overline{n}|}}{dx} = [\mu(x) + \delta]\bar{A}^1_{x:\overline{n}|} + \mu(x + n)A^1_{x:\overline{n}|} - \mu(x) \qquad x \geq 0.$$

4.22. Solve the differential equation (4.5.1) as follows:
a. Use the integrating factor

$$\exp\left[-\int_y^x (\delta + \mu(z) \, dz\right]$$

to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \exp\left\{-\int_y^x [\delta + \mu(z)] \, dz\right\} dx.$$

b. Use the integrating factor $e^{-\delta x}$ to obtain

$$\bar{A}_y = \int_y^{\infty} \mu(x) \, v^{x-y}(1 - \bar{A}_x) \, dx.$$

Miscellaneous

4.23. Display the actuarial present value of a Double Protection to Age 65 policy that provides a benefit of 2 in the event of death prior to age 65 and a benefit

of 1 after age 65 in symbols of Table 4.3.1. Assume benefits are paid at the end of the year of death.

4.24. A policy is issued at age 20 with the following graded scale of death benefits payable at the moment of death.

Age	Death Benefits
20	1 000
21	2 000
22	4 000
23	6 000
24	8 000
25–40	10 000
41 and over	50 000

$\min A_x = v^m {}_m p_x A^1_{x+m:\overline{n}|}$

$A_{x+m:\overline{n}|} = A_{x+m} - v^n {}_n p_{x+m} \cdot A_{x+m+n}$.

Calculate the actuarial present value on the basis of the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$. [Hint: A backward recursion formula for the actuarial present value will include a function $c(x)$ that incorporates the death benefit scale.]

4.25. a. Determine whether or not a constant increase in the force of mortality has the same effect on A_x as the same increase in the force interest.

b. Show that if the single probability of death q_{x+n} is increased to $q_{x+n} + c$, then A_x will be increased by

$$c\, v^{n+1} {}_n p_x \, (1 - A_{x+n+1}).$$

4.26. The actuarial present value for a modified pure endowment of 1,000 issued at age x for n years is 700 with a death benefit equal to the actuarial present value in event of death during the n-year period and is 650 with no death benefit.

a. Calculate the actuarial present value for a modified pure endowment of 1,000 issued at age x for n years if $100k\%$ of the actuarial present value is to be paid at death during the period.

b. For the modified pure endowment in (a), express the variance of the present value at policy issue in terms of actuarial present values for pure endowments and term insurances.

4.27. An appliance manufacturer sells his product with a 5-year warranty promising the return of cash equal to the pro rata share of the initial purchase price for failure within 5 years. For example, if failure is reported 3-3/4 years following purchase, 25% of the purchase price will be returned. From statistical studies, the probability of failure of a new product during the first year is estimated to be 0.2, in each of the second, third, and fourth years 0.1, and in the fifth year 0.2.

a. Assuming that failures are reported uniformly within each year since purchase, determine the fraction of the purchase price that equals the actuarial present value for this warranty. Assume $i = 0.10$.

b. If the warranted return is the reduction on the purchase price of a new product with a 5-year warranty, would the answer to (a) change?

4.28. Assume that $T(x)$ has a p.d.f. given by

$$f_{T(x)}(t) = \frac{2}{10\sqrt{2\pi}} e^{-t^2/200} \qquad t > 0$$

and $\delta = 0.05$.
Show:
a. $\bar{A}_x = 2e^{0.125}[1 - \Phi(0.5)] = 0.6992$
b. ${}^2\bar{A}_x = 2e^{-0.5}[1 - \Phi(1)] = 0.5232$
c. $\text{Var}(Z) = 0.0343$, where $Z = v^T$
d. $\xi_Z^{0.5} = 0.7076$ [Hint: Use Figure 4.2.1]
e. $v^{\mathring{e}_x} = 0.6710 < 0.6992 = \bar{A}_x$.

4.29. Generalize Exercise 4.28(e) by showing that if $\delta > 0$, then

$$v^{\mathring{e}_x} \leq \bar{A}_x.$$

[Hint: Use Jensen's Inequality in Section 1.3 when $u''(x) > 0$.]

4.30. For a whole life insurance the benefit amount, b_t, is 0 or 1 for each $t \geq 0$. For calculations at force of interest δ_t:
a. Express the discount function in terms of δ_t.
b. Express the present-value random variable, Z, in terms of T.
c. Show that Z/@force of interest δ_t equals Z@force of interest $j\delta_t$.

Computing Exercises

4.31. Augment your Illustrative Life Table to include input of a constant interest rate, i, and a payment frequency, m. It will be helpful for the equivalent rates δ, $d^{(m)}$, $i^{(m)}$, d, and v to be shown in your Illustrative Life Table output.

4.32. Develop a set of values of $\ddot{a}_{\overline{n}|}$, $n = 1, 2, \ldots, 100$ at $i = 0.06$ by using the forward recursion formula (3.5.20). [Hint: Review Exercise 3.25 or use $\ddot{a}_{\overline{n+1}|} = 1 + v\ddot{a}_{\overline{n}|}$.]

4.33. a. Using the backward recursion formula of (4.3.10) in your Illustrative Life Table and the appropriate starting value, calculate $1{,}000A_x$ for $x = 13$ to 140 for an interest rate of 0.06.
b. Compare your values to those in Appendix 2A.

4.34. a. Use (4.3.3) and your Illustrative Life to determine $A^1_{20:\overline{20}|}$ and ${}^2A^1_{20:\overline{20}|}$ at $i = 0.05$.
b. What is the variance for the present-value random variable for a 20-year term insurance with benefit amount 100,000 issued to (20)?

4.35. a. By an algebraic or probabilistic argument, verify the following backward recursion formula of an n-year term insurance with a unit benefit:

$$A^1_{x:\overline{n}|} = vq_x - v^{n+1}{}_np_x\, q_{x+n} + vp_x\, A^1_{x+1:\overline{n}|}.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with $i = 0.06$ to calculate the actuarial present value of a 10-year term insurance issued at ages $x = 13, \ldots, 130$.

4.36. a. Use recursion relation (g) at the end of Section 4.3 and your Illustrative Life Table to calculate $(IA)_{28}$ at $i = 0.06$.

b. Modify the recursion relation of part (a) to obtain one for $(I\bar{A})_x$ and determine a starting value for it.

c. Modify the recursion relation of part (b) to obtain one for $(\bar{I}\bar{A})_x$ and determine a starting value for it.

d. Make the recursion relations of parts (b) and (c) specific to the assumption of a uniform distribution of deaths over each year of age.

4.37. Use your Illustrative Life Table to verify the numerical solutions to parts (a) and (b) of Example 4.2.4. [Hint: Set $B = 0.00$ and $A = 0.04$ in your Makeham law parameters, $i = e^{0.10} - 1$, and use recursion formula (d) at the end of Section 4.3. Remember that the insurance in Example 4.2.4 is payable at the moment of death.]

4.38. a. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit endowment insurance to age y with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{y-x}|} = \bar{A}^1_{x:\overline{1}|} + vp_x\, \bar{A}_{x+1:\overline{y-(x+1)}|} \qquad x = 0, 1, \ldots, y-1.$$

b. Determine an appropriate starting value for use with this formula.

c. Use your Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ to calculate the actuarial present value of a unit benefit endowment insurance to age 65 with the death benefit payable at the moment of death for issue ages $x = 13, \ldots, 64$.

d. By an algebraic or probabilistic argument, verify the following backward recursion formula for the actuarial present value of a unit benefit n-year endowment insurance with the death benefit payable at the moment of death:

$$\bar{A}_{x:\overline{n}|} = \bar{A}^1_{x:\overline{1}|} + v^n\,{}_np_x\,(1 - vp_{x+n} - \bar{A}^1_{x+1:\overline{n}|}) + vp_x\, \bar{A}_{x+1:\overline{n}|}$$

$$x = 0, 1, \ldots, w-1.$$

4.39. Let Z be the present-value random variable for a 100,000 unit 20-year endowment insurance with the death benefit payable at the moment of death. Use your Illustrative Life Table to calculate the mean and the variance of Z on the basis of a Makeham law with $A = 0.001$, $B = 0.00001$, $c = 1.10$, and $\delta = 0.05$.



LIFE ANNUITIES

5.1 Introduction

In the preceding chapter we studied payments contingent on death, as provided by various forms of life insurances. In this chapter we study payments contingent on survival, as provided by various forms of life annuities. A ***life annuity*** is a series of payments made continuously or at equal intervals (such as months, quarters, years) while a given life survives. It may be temporary, that is, limited to a given term of years, or it may be payable for the whole of life. The payment intervals may commence immediately or, alternatively, the annuity may be deferred. Payments may be due at the beginnings of the payment intervals (***annuities-due***) or at the ends of such intervals (***annuities-immediate***).

Through the study of ***annuities-certain*** in the theory of interest, the reader already has a knowledge of annuity terminology, notation, and theory. Life annuity theory is analogous but brings in survival as a condition for payment. This condition has been encountered in Chapter 4 in connection with pure endowments and the maturity payments under endowment insurances.

Life annuities play a major role in life insurance operations. As we see in the next chapter, life insurances are usually purchased by a life annuity of premiums rather than by a single premium. The amount payable at the time of claim may be converted through a settlement option into some form of life annuity for the beneficiary. Some types of life insurance carry this concept even further and, instead of featuring a lump sum payable on death, provide stated forms of income benefits. Thus, for example, there may be a monthly income payable to a surviving spouse or to a retired insured.

Annuities are even more central in pension systems. In fact, a retirement plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) by some form of temporary annuity of contributions during active service. The temporary annuity may consist of varying contributions, and valuation of it may take into account not only interest and mortality, but other factors such as salary increases and the termination of participation for reasons other than death.

Life annuities also have a role in disability and workers' compensation insurances. In the case of disability insurance, termination of the annuity benefit by reason of recovery of the disabled insured may need to be considered. For surviving spouse benefits under workers' compensation, remarriage may terminate the annuity.

We proceed in this chapter as we did in Chapter 4 and express the present value of benefits to be received by the annuitant as a function of T, the annuitant's future-lifetime random variable. It then will be possible to study properties of the distribution of this financial value random variable. The expectation, still called the actuarial present value, can be evaluated in an alternative way using either integration by parts or summation by parts depending, respectively, on whether a continuous or discrete set of payments is being evaluated. The results of this process have a useful interpretation and lead to an alternative method of obtaining actuarial present values called the ***current payment technique.***

As in the preceding chapter on life insurance, unless otherwise stated we assume a constant effective annual rate of interest i (or the equivalent constant force of interest δ). We also assume aggregate mortality for most of the development in this chapter and indicate those situations where a select mortality assumption makes a major difference.

In most applications of the theory developed in this chapter, annuity payments continue while a human life remains in a particular status. However, the possible applications of the theory are much wider. It may be applied to any set of periodic payments where the payments are not made with certainty. Examples of some of these applications are seen in later chapters dealing with multiple lives or multiple causes of decrement.

5.2 Continuous Life Annuities

We start with annuities payable continuously at the rate of 1 per year. This is of course an abstraction but makes use of familiar mathematical tools and as a practical matter closely approximates annuities payable on a monthly basis. A ***whole life annuity*** provides for payments until death. Hence, the present value of payments to be made is $Y = \bar{a}_{\overline{T}|}$ for all $T \geq 0$ where T is the future lifetime of (x). The distribution function of Y can be obtained from that for T as follows:

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) = \Pr(\bar{a}_{\overline{T}|} \leq y) = \Pr(1 - v^T \leq \delta y) \\ &= \Pr(v^T \geq 1 - \delta y) = \Pr\left[T \leq \frac{-\log(1 - \delta y)}{\delta}\right] \\ &= F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right) \quad \text{for } 0 < y < \frac{1}{\delta}. \end{aligned} \tag{5.2.1}$$

From this we obtain the probability density function for Y as

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right)$$

$$= \frac{f_T([-\log(1 - \delta y)] / \delta)}{1 - \delta y} \quad \text{for } 0 < y < \frac{1}{\delta}. \tag{5.2.2}$$

The distribution function for Y depends on the distribution of T but would resemble that shown at the end of this section in Figure 5.2.1(a).

The actuarial present value for a continuous whole life annuity is denoted by $\bar{a}_x$ where the post fixed subscript, x, indicates that the annuity ceases at the death of (x) and that the distribution of $T(x)$ may depend on information available at age x. Under aggregate mortality the p.d.f. of T is ${}_tp_x\ \mu(x + t)$, and the actuarial present value can be calculated by

$$\bar{a}_x = \mathrm{E}[Y] = \int_0^\infty \bar{a}_{\overline{t}|}\ {}_tp_x\ \mu(x + t)\ dt. \tag{5.2.3}$$

Using integration-by-parts with $f(t) = \bar{a}_{\overline{t}|}$, $dg(t) = {}_tp_x\ \mu(x + t)\ dt$, $g(t) = -{}_tp_x$, and $df(t) = v^t\ dt$, we obtain

$$\bar{a}_x = \int_0^\infty v^t\ {}_tp_x\ dt = \int_0^\infty {}_tE_x\ dt. \tag{5.2.4}$$

This integral can be considered as involving a momentary payment of 1 dt made at time t, discounted at interest back to time zero by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t. This is the current payment form of the actuarial present value for the whole life annuity. In general, the current payment technique for determining an actuarial present value for an annuity gives

$$\mathrm{APV} = \int_0^\infty v^t\ \Pr[\text{payments are being made at time } t]$$

$$\times\ [\text{Payment rate at time } t]\ dt. \tag{5.2.5}$$

Let us rewrite (5.2.4) by splitting off that portion of the integral involving t values from 0 to 1. Thus

$$\bar{a}_x = \int_0^1 v^t\ {}_tp_x\ dt + \int_1^\infty v^t\ {}_tp_x\ dt$$

$$= \bar{a}_{x:\overline{1}|} + v\ p_x \int_0^\infty v^s\ {}_sp_{x+1}\ ds$$

$$= \bar{a}_{x:\overline{1}|} + v\ p_x\ \bar{a}_{x+1}. \tag{5.2.6}$$

The actuarial present-value symbol, $\bar{a}_{x:\overline{1}|}$, used above is introduced below in (5.2.11). Expression (5.2.6) is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Section 4.3. Here $u(x) = \bar{a}_x$, $c(x) = \bar{a}_{x:\overline{1}|}$, and $d(x) = v\ p_x$. The initial value to use for the whole life annuity is $\bar{a}_\omega = 0$. There are several ways to evaluate the $c(x)$ term. A simple approach is to use a trapezoid approximation for the integral

$$\bar{a}_{x:\overline{1}|} = \int_0^1 v^t \, {}_tp_x \, dt = \frac{1 + v\, p_x}{2}.$$

Another approach, based on the assumption of a uniform distribution of deaths within each year of age, is examined in Section 5.4.

A relationship, familiar from compound interest theory, is that

$$1 = \delta \, \bar{a}_{\overline{t}|} + v^t.$$

This can be interpreted as indicating that a unit invested now will produce annual interest of δ payable continuously for t years at which point interest ceases and the investment is repaid. This relationship holds for all values of t and thus is true for the random variable T:

$$1 = \delta \, \bar{a}_{\overline{T}|} + v^T. \tag{5.2.7}$$

Then, taking expectations, we obtain

$$1 = \delta \, \bar{a}_x + \bar{A}_x. \tag{5.2.8}$$

This is subject to the same kind of interpretation as above. A unit invested now will produce annual interest of δ payable continuously for as long as (x) survives, and, at the time of death, interest ceases and the investment of 1 is repaid.

To measure, on the basis of the assumptions in our model, the mortality risk in a continuous life annuity, we are interested in $\text{Var}(\bar{a}_{\overline{T}|})$. We determine

$$\begin{aligned} \text{Var}(\bar{a}_{\overline{T}|}) &= \text{Var}\left(\frac{1 - v^T}{\delta}\right) \\ &= \frac{\text{Var}(v^T)}{\delta^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}. \end{aligned} \tag{5.2.9}$$

Further, we can observe that since $1 = \delta \, \bar{a}_{\overline{T}|} + v^T$, $\text{Var}(\delta \, \bar{a}_{\overline{T}|} + v^T) = 0$. Thus there is no mortality risk for the combination of a continuous life annuity of δ per year and a life insurance of 1 payable at the moment of death.

Example 5.2.1

Under the assumptions of a constant force of mortality, μ, and of a constant force of interest, δ, evaluate

a. $\bar{a}_x = \text{E}[\bar{a}_{\overline{T}|}]$

b. $\text{Var}(\bar{a}_{\overline{T}|})$

c. The probability that $\bar{a}_{\overline{T}|}$ will exceed $\bar{a}_x$.

Solution:

a. $$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt = \int_0^\infty e^{-\delta t} e^{-\mu t} \, dt = \frac{1}{\delta + \mu}.$$

b. $\bar{A}_x = 1 - \delta\,\bar{a}_x = \dfrac{\mu}{\delta + \mu}$

$^2\bar{A}_x = \dfrac{\mu}{2\delta + \mu}$ by the rule of moments

$$\text{Var}(\bar{a}_{\overline{T}|}) = \frac{1}{\delta^2}\left[\frac{\mu}{2\delta + \mu} - \left(\frac{\mu}{\delta + \mu}\right)^2\right] = \frac{\mu}{(2\delta + \mu)(\delta + \mu)^2}.$$

c. $$\Pr(\bar{a}_{\overline{T}|} > \bar{a}_x) = \Pr\left(\frac{1 - v^T}{\delta} > \bar{a}_x\right) = \Pr\left[T > -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)\right]$$

$$= {}_{t_0}p_x \qquad \text{where } t_0 = -\frac{1}{\delta}\log\left(\frac{\mu}{\delta + \mu}\right)$$

$$= \left(\frac{\mu}{\delta + \mu}\right)^{\mu/\delta}.$$

▼

We now turn to temporary and deferred life annuities. The present value of a benefits random variable for an ***n-year temporary life annuity*** of 1 per year, payable continuously while (x) survives during the next n years, is

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.10}$$

The distribution of Y in this case is a mixed distribution. In particular, the maximum value of Y is limited to $\bar{a}_{\overline{n}|}$, and there is a positive probability associated with $\bar{a}_{\overline{n}|}$ of $\Pr(T \ge n) = {}_np_x$. A typical distribution function for this random variable is illustrated in Figure 5.2.1(b).

The actuarial present value of an n-year temporary life annuity is denoted by $\bar{a}_{x:\overline{n}|}$ and equals

$$\bar{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \int_0^n \bar{a}_{\overline{t}|}\,{}_tp_x\,\mu(x + t)\,dt + \bar{a}_{\overline{n}|}\,{}_np_x. \tag{5.2.11}$$

Integration by parts gives

$$\bar{a}_{x:\overline{n}|} = \int_0^n v^t\,{}_tp_x\,dt. \tag{5.2.12}$$

This is the current payment integral for the actuarial present value for the n-year temporary annuity. It can be considered as involving a momentary payment 1 dt made at time t, discounted at interest back to time 0 by multiplying by v^t and further multiplied by ${}_tp_x$ to reflect the probability that a payment is made at time t for times up to time n. No payments are to be made after time n so the probability of such payments is 0.

The same recursion formula as indicated for (5.2.6) applies here with $u(x) = \bar{a}_{x:\overline{y-x}|}$ and the same $c(x)$ function which we now recognize as $\bar{a}_{x:\overline{1}|}$. We use here $n = y - x$. The only thing that needs to be changed is the initial value, for which we use $u(y) = \bar{a}_{y:\overline{0}|} = 0$. Another form of a recursion formula for a temporary annuity with the n-year period fixed is examined in Exercise 5.7.

Returning to (5.2.10) we note that

$$Y = \begin{cases} \bar{a}_{\overline{T}|} = \dfrac{1 - Z}{\delta} & 0 \le T < n \\ \bar{a}_{\overline{n}|} = \dfrac{1 - Z}{\delta} & T \ge n \end{cases} \tag{5.2.13}$$

where

$$Z = \begin{cases} v^T & 0 \le T < n \\ v^n & T \ge n. \end{cases} \tag{5.2.14}$$

In (5.2.14), Z is the present-value random variable for an n-year endowment insurance. Hence

$$\mathrm{E}[Y] = \bar{a}_{x:\overline{n}|} = \mathrm{E}\left[\frac{1 - Z}{\delta}\right] = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta} \tag{5.2.15}$$

and

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{\delta^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - \bar{A}_{x:\overline{n}|}^2}{\delta^2}. \tag{5.2.16}$$

In terms of annuity values, (5.2.16) becomes

$$\mathrm{Var}(Y) = \frac{1 - 2\delta\,{}^2\bar{a}_{x:\overline{n}|} - (1 - \delta\,\bar{a}_{x:\overline{n}|})^2}{\delta^2}$$

$$= \frac{2}{\delta}\left(\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}\right) - \left(\bar{a}_{x:\overline{n}|}\right)^2.$$

The analysis for an **n-*year deferred whole life annuity*** is similar. The present-value random variable Y is defined as

$$Y = \begin{cases} 0 = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{T}|} & 0 \le T < n \\ v^n\,\bar{a}_{\overline{T-n}|} = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n. \end{cases} \tag{5.2.17}$$

Here the random variable Y can take on a value no larger than $(1/\delta) - \bar{a}_{\overline{n}|} = v^n/\delta$, and the probability that it takes on a zero value is $\Pr(T \le n) = {}_nq_x$. A typical distribution function is illustrated in Figure 5.2.1(c).

Then,

$${}_{n|}\bar{a}_x = \mathrm{E}[Y] = \int_n^\infty v^n\,\bar{a}_{\overline{t-n}|}\,{}_tp_x\,\mu(x + t)\,dt$$

$$= \int_0^\infty v^n\,\bar{a}_{\overline{s}|}\,{}_{n+s}p_x\,\mu(x + n + s)\,ds$$

$$= v^n\,{}_np_x \int_0^\infty \bar{a}_{\overline{s}|}\,{}_sp_{x+n}\,\mu(x + n + s)\,ds$$

which shows that

$${}_{n|}\bar{a}_x = {}_nE_x\,\bar{a}_{x+n}. \tag{5.2.18}$$

An alternative development would be to note that, from the definitions of Y,

$$\begin{pmatrix} Y \text{ for an } n\text{-year deferred} \\ \text{whole life annuity} \end{pmatrix} = \begin{pmatrix} Y \text{ for a whole life annuity} \end{pmatrix} - \begin{pmatrix} Y \text{ for an } n\text{-year temporary} \\ \text{life annuity} \end{pmatrix}.$$

Taking expectations gives

$$ {}_{n|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|}. \tag{5.2.19}$$

Integration by parts can be employed to verify the result given by the current payment technique. Since the annuity will be paying after time n if x survives, the actuarial present value can be written as

$$ {}_{n|}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x \, dt = \int_n^\infty {}_tE_x \, dt. \tag{5.2.20}$$

To develop the backward recursion formula for deferred annuities with $n = y - x > 1$, we note that we have no term corresponding to the integral for t values between 0 and 1. Thus, for $u(x) = {}_{y-x|}\bar{a}_x$ at ages less than y, $c(x) = 0$, and $d(x) = v\,p_x$. For a starting value we would use $u(y) = \bar{a}_y$.

One way to calculate the variance of Y for the deferred annuity is the following:

$$\begin{aligned} \text{Var}(Y) &= \int_n^\infty v^{2n} \, (\bar{a}_{\overline{t-n}|})^2 \, {}_tp_x \, \mu(x+t) \, dt - ({}_{n|}\bar{a}_x)^2 \\ &= v^{2n} \, {}_np_x \int_0^\infty (\bar{a}_{\overline{s}|})^2 \, {}_sp_{x+n} \, \mu(x+n+s) \, ds - ({}_{n|}\bar{a}_x)^2, \end{aligned}$$

and using integration by parts,

$$\begin{aligned} &= v^{2n} \, {}_np_x \int_0^\infty 2\bar{a}_{\overline{s}|} \, v^s \, {}_sp_{x+n} \, ds - ({}_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} \, v^{2n} \, {}_np_x \int_0^\infty (v^s - v^{2s}) \, {}_sp_{x+n} \, ds - ({}_{n|}\bar{a}_x)^2 \\ &= \frac{2}{\delta} \, v^{2n} \, {}_np_x \, (\bar{a}_{x-n} - {}^2\bar{a}_{x+n}) - ({}_{n|}\bar{a}_x)^2. \end{aligned} \tag{5.2.21}$$

For an alternative development of this formula, see Exercise 5.37.

We now turn to analysis of an **n-*year certain and life annuity.*** This is a whole life annuity with a guarantee of payments for the first n years. The present value of annuity payments is

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & T \le n \\ \bar{a}_{\overline{T}|} & T > n. \end{cases} \tag{5.2.22}$$

A typical distribution function is shown in Figure 5.2.1(d), which reflects the mixed nature of the distribution and the minimum value and upper bound of Y, which are $\bar{a}_{\overline{n}|}$ and $1/\delta$, respectively.

The actuarial present value is denoted by $\bar{a}_{\overline{x:n}|}$. This symbol is adopted to indicate that payments continue until $\max[T(x), n]$:

$$\begin{aligned}\bar{a}_{\overline{x:n}|} = \mathrm{E}[Y] &= \int_0^n \bar{a}_{\overline{n-t}|}\, {}_tp_x\, \mu(x+t)\, dt \\ &\quad + \int_n^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt \\ &= {}_nq_x\, \bar{a}_{\overline{n}|} + \int_n^\infty \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt. \end{aligned} \tag{5.2.23}$$

Integration by parts can be used to obtain

$$\bar{a}_{\overline{x:n}|} = \bar{a}_{\overline{n}|} + \int_n^\infty v^t\, {}_tp_x\, dt. \tag{5.2.24}$$

This is the current payment form for the actuarial present value, since at times 0 to n payment is certain, whereas for times greater than n payment is made if (x) survives.

Further insight can be obtained by rewriting Y as

$$Y = \begin{cases} \bar{a}_{\overline{n}|} + 0 & T \le n \\ \bar{a}_{\overline{n}|} + (\bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}) & T > n. \end{cases}$$

Here Y is the sum of a constant $\bar{a}_{\overline{n}|}$ and the random variable for the n-year deferred annuity. Thus,

$$\begin{aligned}\bar{a}_{\overline{x:n}|} &= \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x \\ &= \bar{a}_{\overline{n}|} + {}_nE_x\, \bar{a}_{x+n} \qquad \text{by (5.2.18)} \\ &= \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|}) \qquad \text{by (5.2.19).} \end{aligned} \tag{5.2.25}$$

Furthermore, since $\mathrm{Var}(Y - \bar{a}_{\overline{n}|}) = \mathrm{Var}(Y)$, the variance for the n-year certain and life annuity is the same as that of the n-year deferred annuity given by (5.2.21).

A backward recursion for $\bar{a}_{\overline{x:n}|}$ with a fixed n-year certain period is examined in Exercise 5.9.

Analogous to the function

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^{n-t}\, dt$$

in the theory of interest, we have for life annuities

$$\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_nE_x} = \int_0^n \frac{1}{{}_{n-t}E_{x+t}}\, dt, \tag{5.2.26}$$

representing the actuarial accumulated value at the end of the term of an n-year temporary life annuity of 1 per year payable continuously while (x) survives.

Such accumulated value, which is often said to have been accumulated under (or with the benefit of) interest and survivorship, is available at age $x + n$ if (x) survives.

We obtain an expression for $d\bar{a}_x / dx$ by differentiating the integral in (5.2.4), assuming that the probabilities are derived from an aggregate table:

$$\frac{d}{dx}\bar{a}_x = \int_0^\infty v^t \left(\frac{\partial}{\partial x}\,{}_tp_x\right) dt = \int_0^\infty v^t\,{}_tp_x\,[\mu(x) - \mu(x+t)]\,dt$$
$$= \mu(x)\,\bar{a}_x - \bar{A}_x = \mu(x)\,\bar{a}_x - (1 - \delta\,\bar{a}_x).$$

Therefore,

$$\frac{d}{dx}\bar{a}_x = [\mu(x) + \delta]\,\bar{a}_x - 1. \tag{5.2.27}$$

The interpretation of (5.2.27) is that the actuarial present value changes at a rate that is the sum of the rate of interest income $\delta\,\bar{a}_x$ and the rate of survivorship benefit $\mu(x)\,\bar{a}_x$, less the rate of payment outgo.

Example 5.2.2

Assuming that probabilities come from an aggregate table, obtain formulas for

a. $\dfrac{\partial}{\partial x}\bar{a}_{x:\overline{n}|}$ b. $\dfrac{\partial}{\partial n}\,{}_{n|}\bar{a}_x$.

Solution:

a. Proceeding as in the development of (5.2.27), we obtain

$$\frac{\partial}{\partial x}\bar{a}_{x:\overline{n}|} = \mu(x)\,\bar{a}_{x:\overline{n}|} - \bar{A}^1_{x:\overline{n}|}$$
$$= \mu(x)\,\bar{a}_{x:\overline{n}|} - (1 - \delta\,\bar{a}_{x:\overline{n}|} - {}_nE_x)$$
$$= [\mu(x) + \delta]\,\bar{a}_{x:\overline{n}|} - (1 - {}_nE_x).$$

b. $\dfrac{\partial}{\partial n}\,{}_{n|}\bar{a}_x = \dfrac{\partial}{\partial n}\displaystyle\int_n^\infty v^t\,{}_tp_x\,dt = -v^n\,{}_np_x.$ ▼

Table 5.2.1 summarizes concepts for continuous life annuities.

Figure 5.2.1 shows typical distribution functions for the several types of continuous life annuities studied in this section. Limiting values and points of discontinuities are indicated on one or both axes.

When $F_Y(0) = 0$, $\mathrm{E}[Y] = \int_0^\infty [1 - F_Y(y)]\,dy$, the actuarial present value of Y can be visualized as the area above the graph of $z = F_Y(y)$, below $z = 1$, and to the right of the line $y = 0$. This interpretation can provide a bridge between the actuarial

Summary of Continuous Life Annuities (Annuity of 1 per Annum Payable Continuously)

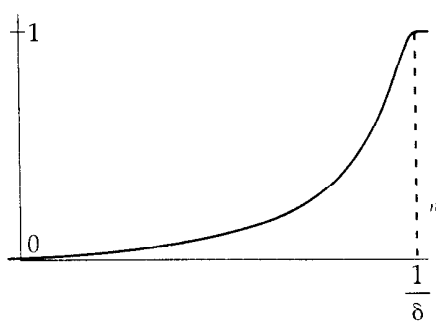
Annuity Name	Present-Value Random Variable Y	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity	$\bar{a}_{\overline{T}\rceil}$ $\quad T \geq 0$	$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$
n-Year Temporary Life Annuity	$\begin{cases} \bar{a}_{\overline{T}\rceil} & 0 \leq T < n \\ \bar{a}_{\overline{n}\rceil} & T \geq n \end{cases}$	$\bar{a}_{x:\overline{n}\rceil} = \int_0^n v^t \, {}_tp_x \, dt$
n-Year Deferred Whole Life Annuity	$\begin{cases} 0 & 0 \leq T < n \\ \bar{a}_{\overline{T}\rceil} - \bar{a}_{\overline{n}\rceil} & T \geq n \end{cases}$	${}_{n\rceil}\bar{a}_x = \int_n^\infty v^t \, {}_tp_x \, dt$
n-Year Certain and Life Annuity	$\begin{cases} \bar{a}_{\overline{n}\rceil} & 0 \leq T < n \\ \bar{a}_{\overline{T}\rceil} & T \geq n \end{cases}$	$\bar{a}_{x:\overline{n}\rceil} = \bar{a}_{\overline{n}\rceil} + \int_n^\infty v^t \, {}_tp_x \, dt$

Additional relations are

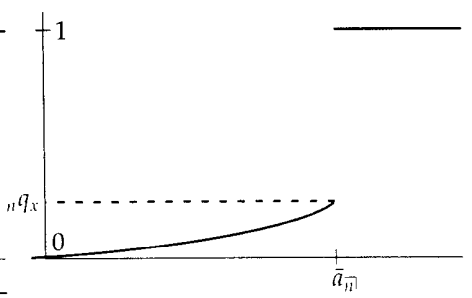
- $1 = \delta \, \bar{a}_x + \bar{A}_x$
- $1 = \delta \, \bar{a}_{x:\overline{n}\rceil} + \bar{A}_{x:\overline{n}\rceil}$
- ${}_{n\rceil}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}\rceil}$
- $\bar{s}_{x:\overline{n}\rceil} = \dfrac{\bar{a}_{x:\overline{n}\rceil}}{{}_nE_x} = \displaystyle\int_0^n (1+i)^{n-t} \frac{l_{x+t}}{l_{x+n}} \, dt.$

Typical Distribution Functions for the Present-Value Random Variables, Y

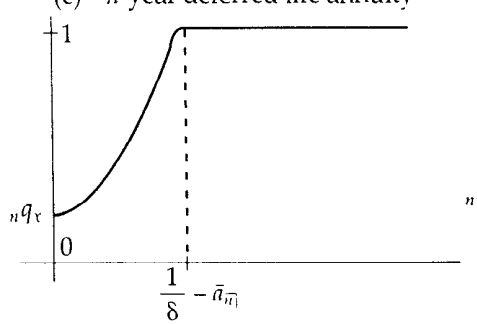
(a) Whole life annuity



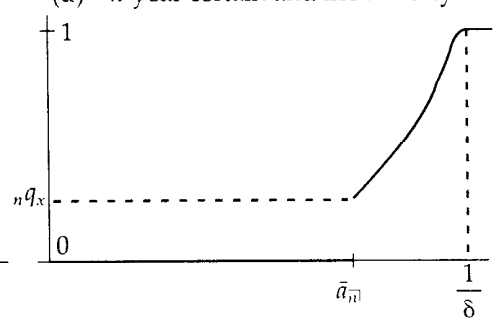
(b) n-year temporary life annuity



(c) n-year deferred life annuity



(d) n-year certain and life annuity



present value as evaluated from the definition of the random variable and the current payment form for the actuarial present value.

5.3 Discrete Life Annuities

The theory of discrete life annuities is analogous, step-by-step, to the theory of continuous life annuities, with integrals replaced by sums, integrands by summands, and differentials by differences. For continuous annuities there was no distinction between payments at the beginning of payment intervals or at the ends, that is, between annuities-due and annuities-immediate. For discrete annuities, the distinction is meaningful, and we start with annuities-due as they have a more prominent role in actuarial applications. For example, most individual life insurances are purchased by an annuity-due of periodic premiums.

We consider an annuity that pays a unit amount at the beginning of each year that the annuitant (x) survives. In the nomenclature this is called a ***whole life annuity-due.*** The present-value random variable, Y, for such an annuity, is given by $Y = \ddot{a}_{\overline{K+1}|}$ where the random variable K is the curtate-future-lifetime of (x). The possible values of this random variable are a discrete set of values ranging from $\ddot{a}_{\overline{1}|} = 1$ to $\ddot{a}_{\overline{\omega-x}|}$, a value which is less than $1/d$. The probability associated with the value $\ddot{a}_{\overline{k+1}|}$ is $\Pr(K = k) = {}_kp_x\, q_{x+k}$.

Let us now consider $\ddot{a}_x$, the actuarial present value of the annuity:

$$\ddot{a}_x = \mathrm{E}[Y] = \mathrm{E}[\ddot{a}_{\overline{K+1}|}]$$

$$= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_kp_x\, q_{x+k}, \tag{5.3.1}$$

since $\Pr(K = k) = {}_kp_x\, q_{x+k}$. By summation-by-parts (see Appendix 5) with $\Delta f(k) = {}_kp_x\, q_{x+k} = {}_kp_x - {}_{k+1}p_x$ and $g(k) = \ddot{a}_{\overline{k+1}|}$ and use of the relations

$$\Delta g(k) = \Delta \ddot{a}_{\overline{k+1}|} = v^{k+1} \text{ and } f(k) = -{}_kp_x,$$

(5.3.1) converts to

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1}\, {}_{k+1}p_x \tag{5.3.2}$$

$$= \sum_{k=0}^{\infty} v^k\, {}_kp_x. \tag{5.3.3}$$

The expression (5.3.3) is the current payment form of the actuarial present value for a whole life annuity-due where the ${}_kp_x$ term is the probability of a payment of size 1 being made at time k.

Starting with the sum in (5.3.2) above we have

$$\ddot{a}_x = 1 + \sum_{k=0}^{\infty} v^{k+1}\, {}_{k+1}p_x = 1 + v\, p_x \sum_{k=0}^{\infty} v^k\, {}_kp_{x+1}$$

$$= 1 + v\, p_x\, \ddot{a}_{x+1}. \tag{5.3.4}$$

This expression is an example of the backward recursion formula first observed in Section 3.5 and explored more fully in Sections 4.3 and 5.2. Here $u(x) = \ddot{a}_x$, $c(x) = 1$, and $d(x) = v\, p_x$. The initial value to use for the whole life annuity is $\ddot{a}_\omega = 0$.

From (5.3.1) we obtain in succession

$$\ddot{a}_x = \mathrm{E}\left[\frac{1 - v^{K+1}}{d}\right]$$

$$= \frac{1 - A_x}{d}, \tag{5.3.5}$$

and

$$\ddot{a}_x = \ddot{a}_{\overline{\infty}|} - \ddot{a}_{\overline{\infty}|}\, A_x, \tag{5.3.6}$$

$$1 = d\ddot{a}_x + A_x. \tag{5.3.7}$$

This should be compared with its continuous counterpart (5.2.8). Formula (5.3.7) indicates that a unit invested now will produce interest-in-advance of d per year while (x) survives plus repayment of the unit at the end of the year of death of (x).

The variance formula is

$$\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \mathrm{Var}\left(\frac{1 - v^{K+1}}{d}\right) = \frac{\mathrm{Var}(v^{K+1})}{d^2}$$

$$= \frac{{}^2A_x - (A_x)^2}{d^2}; \tag{5.3.8}$$

see (5.2.9).

The present-value random variable of an **n-*year temporary life annuity-due*** of 1 per year is

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \le K < n \\ \ddot{a}_{\overline{n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\ddot{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_kp_x\, q_{x+k} + \ddot{a}_{\overline{n}|}\, {}_np_x. \tag{5.3.9}$$

Summation by parts can be used to transform (5.3.9) into

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k\, {}_kp_x, \tag{5.3.10}$$

which is the actuarial present value in the current payment form.

Again separating out the first term and factoring out vp_x, we obtain the backward recursion formula for a temporary annuity-due payable to age $y = x + n$:

$$\ddot{a}_{x:\overline{y-x}|} = 1 + vp_x \, \ddot{a}_{x+1:\overline{y-(x+1)}|}. \tag{5.3.11}$$

This recursive formula for the actuarial present values is the same as (5.3.4) but differs in that here we use an initial value of $\ddot{a}_{y:\overline{0}|} = 0$.

Since $Y = (1 - Z)/d$, where

$$Z = \begin{cases} v^{K+1} & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

is the present-value random variable for a unit of endowment insurance, payable at the end of the year of death or at maturity, we have

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - \mathrm{E}[Z]}{d} = \frac{1 - A_{x:\overline{n}|}}{d}; \tag{5.3.12}$$

see (5.2.15).

Rearrangement of (5.3.12) yields

$$1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}. \tag{5.3.13}$$

To calculate the variance, we can use

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)}{d^2} = \frac{{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2}{d^2}. \tag{5.3.14}$$

For an **n-*year deferred whole life annuity-due*** of 1 payable at the beginning of each year while (x) survives from age $x + n$ onward, the present-value random variable is

$$Y = \begin{cases} 0 & 0 \le K < n \\ {}_{n|}\ddot{a}_{\overline{K+1-n}|} & K \ge n, \end{cases}$$

and its actuarial present value is

$$\mathrm{E}[Y] = {}_{n|}\ddot{a}_x = {}_nE_x \, \ddot{a}_{x+n} \tag{5.3.15}$$

$$= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \tag{5.3.16}$$

$$= \sum_{k=n}^{\infty} v^k \, {}_kp_x; \tag{5.3.17}$$

see (5.2.18)–(5.2.20).

The backward recursion formula for a deferred annuity-due with $n = y - x > 1$ is identical to that for the continuous version in that it uses $c(x) = 0$ and $d(x) = vp_x$. The change is that we use the actuarial present value for an annuity-due, $u(y) = \ddot{a}_y$, for the starting value.

The variance of Y can be developed in a manner completely analogous to that used in formula (5.2.21) and leads to the expression

$$\text{Var}(Y) = \frac{2}{d} v^{2n} {}_n p_x (\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n}) + {}_{n|}{}^2\ddot{a}_x - ({}_{n|}\ddot{a}_x)^2. \tag{5.3.18}$$

We turn now to analysis of an **n-*year certain and life annuity-due.*** This is a life annuity with a guarantee of payments for at least n years. The present value of the annuity payments is

$$Y = \begin{cases} \ddot{a}_{\overline{n}|} & 0 \le K < n \\ \ddot{a}_{\overline{K+1}|} & K \ge n. \end{cases} \tag{5.3.19}$$

Then

$$\ddot{a}_{x:\overline{n}|} = \text{E}[Y] = \ddot{a}_{\overline{n}|} {}_n q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|} {}_k p_x \, q_{x+k}, \tag{5.3.20}$$

and this can be transformed by summation by parts into the current payment version of the actuarial present value

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k {}_k p_x. \tag{5.3.21}$$

This can be written as

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}.$$

The actuarial accumulated value at the end of the term of an n-year temporary life annuity-due of 1 per year, payable while (x) survives, is denoted by $\ddot{s}_{x:\overline{n}|}$. Formulas for this function are

$$\ddot{s}_{x:\overline{n}|} = \frac{\ddot{a}_{x:\overline{n}|}}{{}_n E_x} = \sum_{k=0}^{n-1} \frac{1}{{}_{n-k}E_{x+k}}, \tag{5.3.22}$$

which are analogous to formulas for $\ddot{s}_{\overline{n}|}$ in the theory of interest.

The procedures used above for annuities-due can be adapted for annuities-immediate where payments are made at the ends of the payment periods. For instance, for a ***whole life annuity-immediate,*** the present-value random variable is $Y = a_{\overline{K}|}$. Then,

$$a_x = \text{E}[Y] = \sum_{k=0}^{\infty} {}_k p_x \, q_{x+k} \, a_{\overline{k}|}, \tag{5.3.23}$$

and a summation by parts will give the current payment form of the actuarial present value as

$$a_x = \sum_{k=1}^{\infty} v^k {}_k p_x. \tag{5.3.24}$$

Since Y equals $(1 - v^K)/i = [1 - (1 + i)\, v^{K+1}]/i$, we have, taking expectations,

$$a_x = \text{E}[Y] = \frac{1 - (1 + i)\, A_x}{i}. \tag{5.3.25}$$

This formula can be rewritten as $1 = i a_x + (1 + i)\, A_x$. A comparison of this formula

with (5.3.7) shows that an interest payment of i is made at the end of each year while (x) remains alive and that at the end of the year of death an interest payment of i along with the principal amount of 1 must be paid. This formula has significance for estate taxation. For each unit of an estate, define ia_x as the ***life estate*** and $(1 + i)A_x = 1 - ia_x$ as the ***remainder,*** which, if designated for a qualified charitable organization, is exempt from estate taxation.

The analysis for the other forms of the annuity-immediate is similar. The present-value random variable can be formed in a manner analogous to that for the annuity-due. Formulas for the actuarial present value from the definition and by summation by parts can be obtained. Formulas for the variances of the annuities-immediate in this section can also be obtained.

Example 5.3.1

Find formulas for the expectation and variance of the present-value random variable for the temporary life annuity-immediate.

Solution:

We start with the present-value random variable for an n-year temporary annuity-immediate:

$$Y = \begin{cases} a_{\overline{K}|} = \dfrac{1 - v^K}{i} & 0 \le K < n \\ a_{\overline{n}|} = \dfrac{1 - v^n}{i} & K \ge n. \end{cases}$$

We now introduce two new random variables

$$Z_1 = \begin{cases} (1 + i)v^{K+1} & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

and

$$Z_2 = \begin{cases} 0 & 0 \le K < n \\ v^n & K \ge n \end{cases}$$

so that $Y = (1 - Z_1 - Z_2)/i$ for all K. Now taking expectations, we have

$$\mathrm{E}[Y] = a_{x:\overline{n}|} = \frac{1 - (1 + i)A^1_{x:\overline{n}|} - A^1_{x:\overline{n}|}}{i}.$$

This can be rewritten, following (5.3.13),

$$1 = i\,a_{x:\overline{n}|} + i\,A^1_{x:\overline{n}|} + A_{x:\overline{n}|}.$$

The variance calculation is as follows:

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z_1 + Z_2)}{i^2} = \frac{\mathrm{Var}(Z_1) + 2\,\mathrm{Cov}(Z_1, Z_2) + \mathrm{Var}(Z_2)}{i^2}.$$

Recall $\text{Var}(Z_1) = (1 + i)^2 \, [{}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2]$ and $\text{Var}(Z_2) = v^{2n} \, {}_np_x \, (1 - {}_np_x)$. Since $Z_1 \, Z_2 = 0$ for all K, we have $\text{Cov}(Z_1, Z_2) = -(1 + i) \, A^1_{x:\overline{n}|} \, v^n \, {}_np_x$. Combining, we obtain

$$\text{Var}(Y) = \frac{(1 + i)^2 \, [{}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2] - 2(1 + i) \, A^1_{x:\overline{n}|} \, v^n \, {}_np_x + v^{2n} \, {}_np_x \, (1 - {}_np_x)}{i^2}.$$

▼

Summary of Discrete Life Annuities [Annuity of 1 Per Annum Payable at the Beginning of Each Year (Annuity-Due) or at the End of Each Year (Annuity-Immediate)]

Annuity Name	Present-Value Random Variable Y	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity		
• Due	$\ddot{a}_{\overline{K+1}|} \quad K \geq 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$
• Immediate	$a_{\overline{K}|} \quad K \geq 0$	$a_x = \sum_{k=1}^{\infty} v^k \, {}_kp_x$
n-Year Temporary Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{K+1}|} & 0 \leq K < n \\ \ddot{a}_{\overline{n}|} & K \geq n \end{cases}$	$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{K}|} & 0 \leq K < n \\ a_{\overline{n}|} & K \geq n \end{cases}$	$a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \, {}_kp_x$
n-Year Deferred Whole Life Annuity		
• Due	$\begin{cases} 0 & 0 \leq K < n \\ \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{n}|} & K \geq n \end{cases}$	${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} 0 & 0 \leq K < n \\ a_{\overline{K}|} - a_{\overline{n}|} & K \geq n \end{cases}$	${}_{n|}a_x = \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$
n-Year Certain and Whole Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{n}|} & 0 \leq K < n \\ \ddot{a}_{\overline{K+1}|} & K \geq n \end{cases}$	$\ddot{a}_{\overline{x:n}|} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k \, {}_kp_x$
• Immediate	$\begin{cases} a_{\overline{n}|} & 0 \leq K < n \\ a_{\overline{K}|} & K \geq n \end{cases}$	$a_{\overline{x:n}|} = a_{\overline{n}|} + \sum_{k=n+1}^{\infty} v^k \, {}_kp_x$

Additional relations are

- $1 = d \, \ddot{a}_x + A_x$
- $A_x = v \, \ddot{a}_x - a_x$
- $1 = d \, \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}$
- $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$
- $A^1_{x:\overline{n}|} = v \, \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}$
- $A_{x:\overline{n}|} = v \, \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$
- ${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{a}_{\overline{x:n}|} = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}$
- $\ddot{s}_{x:\overline{n}|} = \dfrac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x} = \sum_{k=0}^{n-1} (1 + i)^{n-k} \, \dfrac{l_{x+k}}{l_{x+n}}.$

5.4 Life Annuities with *m*-thly Payments

In practice, life annuities are often payable on a monthly, quarterly, or semi-annual basis. In International Actuarial Notation, the actuarial present value of a life annuity of 1 per year, payable in installments of 1 / m at the beginning of each m-th of the year while (x) survives, is denoted by $\ddot{a}_x^{(m)}$.

We start the analysis of the distribution of Y, the present value of the life annuity-due, with payments made on an m-thly basis, by expressing Y in terms of the interest rate and the random variables, K and $J = \lfloor (T - K)m \rfloor$. The "$\lfloor\ \rfloor$" in the expression for J denote the greatest integer function so that J is the number of complete m-ths of a year lived in the year of death. For an annuity-due there would be m payments for each of the K complete years and then $J + 1$ payments of 1 / m in the year of death; thus,

$$Y = \sum_{j=0}^{mK+J} \frac{1}{m} v^{j/m} = \ddot{a}_{\overline{K+(J+1)/m}|}^{(m)} = \frac{1 - v^{K+(J+1)/m}}{d^{(m)}}. \tag{5.4.1}$$

The actuarial present value, E[Y], which can be determined using Exercise 4.19, is

$$\mathrm{E}[Y] = \ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}. \tag{5.4.2}$$

The current payment form, which is the sum of the actuarial present value of the set of payments, is

$$\ddot{a}_{\setminus}^{(m)} = \frac{1}{m} \sum_{h=0}^{\infty} v^{h/m}\, {}_{h/m}p_x. \tag{5.4.3}$$

Again using (5.4.1), we obtain

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(v^{K+(J+1)/m})}{(d^{(m)})^2} = \frac{{}^2A_x^{(m)} - (A_x^{(m)})^2}{(d^{(m)})^2}. \tag{5.4.4}$$

It is convenient to use (5.3.7) and (5.4.2) to obtain various relationships between the actuarial present values for m-thly annuities and those with annual payments:

$$1 = d\,\ddot{a}_{\setminus} + A_x = d^{(m)}\,\ddot{a}_{\setminus}^{(m)} + A_x^{(m)}. \tag{5.4.5}$$

These show that an investment of 1 will produce interest-in-advance at the beginning of each interest period and repayment of the unit at the end of the period in which death occurs.

From the right two members of (5.4.5) we obtain

$$\begin{aligned} \ddot{a}_x^{(m)} &= \frac{d}{d^{(m)}}\,\ddot{a}_x - \frac{1}{d^{(m)}}\,(A_x^{(m)} - A_x) \\ &= \ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \ddot{a}_{\overline{1}|}^{(m)}\,(A_x^{(m)} - A_x). \end{aligned} \tag{5.4.6}$$

This can be interpreted as follows: The m-thly payment life annuity is equivalent to a series of 1-year annuities-certain in each year that (x) begins, with cancellation

in the year of death of installments payable beyond the m-th (month, quarter, half-year) of death. The cancellation is accomplished by an m-thly payment perpetuity beginning at the end of the m-th of death less a similar perpetuity beginning at the end of the year of death. Alternatively, from (5.4.2), we can write

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}} = \ddot{a}_{\overline{x}|}^{(m)} - \ddot{a}_{\overline{1}|}^{(m)} A_x^{(m)}, \tag{5.4.7}$$

which is left to the reader to interpret.

Remark:

In the study of interest theory the calculation of the present value of an annuity with payment periods and effective interest periods of different lengths was reduced to the calculation of the present value of an annuity with payment periods and interest periods of equal length in one of two ways. The first was to replace the payments corresponding to an interest period by a single equivalent payment (at the given interest rate) at one end or the other of the interest period. The expression for the m-thly whole life annuity in (5.4.6) is an extension of this method to a set of contingent payments. When calculators with exponentiation keys replaced interest tables, the second method, using the equivalent effective interest rate per payment period, became the preferred way to match payment period and interest period lengths. The extension of this second method to m-thly whole life annuities would be to use an m-thly mortality table along with the equivalent effective interest rate per m-th of a year. Using this, the recursion relations of Section 5.3 could be used to obtain the actuarial present value of the m-thly whole life annuity. An advantage of this second approach is that it frees us to use other assumptions about the distribution of deaths within each year of age, like constant force or Makeham's, in place of the uniform distribution that is central in the discussion below.

Now let us assume that deaths have a uniform distribution in each year of age. This means that S has a uniform distribution on $(0, 1)$ so that J is uniformly distributed on the integers $\{0, 1, \ldots, m-1\}$. Exercise 4.19 shows that this implies

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x = s_{\overline{1}|}^{(m)} A_x,$$

and from (5.4.6) we have

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_x - \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} A_x. \tag{5.4.8}$$

Formula (5.4.8) shows that the value of the m-thly life annuity-due is the difference between

a. The value of an annual life annuity-due with each annual payment sufficient to pay a 1-year annuity certain of $1/m$ at the beginning of each m-th; and

b. The value of an insurance payable at the end of the year of death with the benefit equal to the coefficient of A_x. It can be shown that, under the assumption of uniform distribution of deaths in each year of age, this coefficient is the actuarial accumulated value of those payments of 1 / m for the m-ths after death. See the bracketed expression in Exercise 5.15.

By substituting $1 - d\,\ddot{a}_x$ for A_x, in (5.4.8) and noting that $d^{(m)}\,\ddot{a}^{(m)}_{\overline{1}|} = d$, we obtain a formula involving only annuity functions, namely,

$$\ddot{a}^{(m)}_x = \frac{1 - s^{(m)}_{\overline{1}|}\,(1 - d\,\ddot{a}_x)}{d^{(m)}}$$

$$= s^{(m)}_{\overline{1}|}\,\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_x - \frac{(s^{(m)}_{\overline{1}|} - 1)}{d^{(m)}}. \tag{5.4.9}$$

An alternative, widely used, formula is

$$\ddot{a}^{(m)}_x = \ddot{a}_x - \frac{m - 1}{2m}. \tag{5.4.10}$$

This result can be obtained by assuming that the function $v^{k+(j/m)}\,{}_{k+(j/m)}p_x$ is linear in j for $j = 0, 1, 2, \ldots, m - 1$. In that case,

$$\sum_{j=0}^{m-1} \frac{1}{m}\,v^{k+(j/m)}\,{}_{k+(j/m)}p_x = \sum_{j=0}^{m-1} \frac{1}{m}\left[\left(1 - \frac{j}{m}\right) v^k\,{}_kp_x + \frac{j}{m}\,v^{k+1}\,{}_{k+1}p_x\right]$$

$$= v^k\,{}_kp_x - (v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x) \sum_{j=0}^{m-1} \frac{j}{m^2}$$

$$= v^k\,{}_kp_x - \frac{m - 1}{2m}\,(v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x).$$

Thus

$$\ddot{a}^{(m)}_x = \sum_{k=0}^{\infty} \sum_{j=0}^{m-1} \frac{1}{m}\,v^{k+(j/m)}\,{}_{k+(j/m)}p_x$$

$$= \sum_{k=0}^{\infty} v^k\,{}_kp_x - \frac{m - 1}{2m} \sum_{k=0}^{\infty} (v^k\,{}_kp_x - v^{k+1}\,{}_{k+1}p_x)$$

$$= \ddot{a}_x - \frac{m - 1}{2m}.$$

Note that this is not the same assumption as that of linearity of ${}_tp_x$, which would follow if a uniform distribution of deaths within each year of age is assumed. Consistent use of the assumption of a uniform distribution of deaths in each year of age assures that relations such as (5.4.5) are satisfied exactly. It has also been observed that formulas derived from (5.4.10) can, for high rates of interest and low rates of mortality, produce distorted annuity values, such as $a^{(m)}_{x:\overline{1}|} > a^{(m)}_{\overline{1}|}$; see Exercise 5.50. For these reasons, (5.4.8) and equivalently (5.4.9) are presented as replacements for the widely used formula (5.4.10).

It is convenient for writing purposes to express (5.4.9) in the form

$$\ddot{a}^{(m)}_x = \alpha(m)\,\ddot{a}_x - \beta(m), \tag{5.4.11}$$

where

$$\alpha(m) = s_{\overline{1}|}^{(m)} \ddot{a}_{\overline{1}|}^{(m)} = \frac{id}{i^{(m)}\, d^{(m)}}, \tag{5.4.12}$$

and

$$\beta(m) = \frac{s_{\overline{1}|}^{(m)} - 1}{d^{(m)}} = \frac{i - i^{(m)}}{i^{(m)}\, d^{(m)}}. \tag{5.4.13}$$

We note that $\alpha(m)$ and $\beta(m)$ depend only on m and the rate of interest and are independent of the year of age. Further, for $m = 1$, (5.4.11) is an identity where $\alpha(1) = 1$ and $\beta(1) = 0$. Also, $\beta(m)$ is the coefficient of the cancellation term in (5.4.8); that is, (5.4.8) can be written as

$$\ddot{a}_x^{(m)} = \ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_x - \beta(m) A_x. \tag{5.4.14}$$

For series expansions of $\alpha(m)$ and $\beta(m)$, see Exercise 5.41.

Example 5.4.1

On the basis of the Illustrative Life Table, with interest at the effective annual rate of 6%, calculate the actuarial present value of a whole life annuity-due of 1,000 per month for a retiree age 65 by (5.4.9) and (5.4.10) and its standard deviation by (5.4.4).

Solution:

Here

$$\alpha(12) = s_{\overline{1}|}^{(12)}\, \ddot{a}_{\overline{1}|}^{(12)} = (1.02721070)(0.97378368) = 1.0002810,$$

$$\beta(12) = \frac{s_{\overline{1}|}^{(12)} - 1}{d^{(12)}} = 0.46811951,$$

$$\frac{11}{24} = 0.45833333.$$

Observe that $\alpha(12) \cong 1$, and $\beta(12)$ is fairly close to the $11/24$ that appears in the traditional approximation.

By the Illustrative Life Table, as defined by (3.7.1), with interest at 6%,

$$\ddot{a}_{65} = 9.89693,$$

$$A_{65} = 1 - d\, \ddot{a}_{65} = 0.4397965.$$

Then, $12{,}000\ddot{a}_{65}^{(12)}$ can be calculated as follows:

by (5.4.11)
$$12{,}000[\alpha(12)\ddot{a}_{65} - \beta(12)]$$
$$= 12{,}000[(1.0002810)(9.89693) - 0.46811951]$$
$$= 113{,}179 \text{ and}$$

by (5.4.10)
$$12{,}000\left(\ddot{a}_{65} - \frac{11}{24}\right) = 113{,}263.$$

The variance of $12{,}000\,Y = 12{,}000(1 - v^{K+(J+1)/12})/d^{(12)}$ is equal to

$$\left(\frac{12{,}000}{d^{(12)}}\right)^2 \mathrm{Var}[v^{K+1}(1+i)^{1-(J+1)/12}]$$

$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \{\mathrm{E}[v^{2(K+1)}(1+i)^{2(1-(J+1)/12)}] - (\mathrm{E}[v^{(K+1)}(1+i)^{(1-(J+1)/12)}])^2\}$$

$$= \left(\frac{12{,}000}{d^{(12)}}\right)^2 \left\{{}^2A_{65}\,\mathrm{E}[(1+i)^{2(1-(J+1)/12)}] - \left(\frac{A_{65}\,i}{i^{(12)}}\right)^2\right\}$$

$$= (206{,}442.14)^2\,[(0.2360299 \times 1.055458268) - (0.4397965 \times 1.02721069)^2]$$

$$= 1{,}919{,}074{,}762.$$

This means that the standard deviation for the present value for the payments to an individual is 43,807 as compared to the actuarial present value of 113,179. ▼

The development starting with random variables can be followed for m-thly temporary and deferred annuities-due. However, if all we seek is formulas for their actuarial present values, we can proceed by starting with (5.4.14). Thus

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_{x} - {}_nE_x\,\ddot{a}^{(m)}_{x+n} \\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x} - \beta(m)A_x - {}_nE_x[\ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x+n} - \beta(m)A_{x+n}] \\ &= \ddot{a}^{(m)}_{\overline{1}|}\,\ddot{a}_{x:\overline{n}|} - \beta(m)A^{1}_{x:\overline{n}|};\end{aligned} \tag{5.4.15}$$

and similarly,

$${}_{n|}\ddot{a}^{(m)}_{x} = \ddot{a}^{(m)}_{\overline{1}|}\,{}_{n|}\ddot{a}_{x} - \beta(m)\,{}_{n|}A_x. \tag{5.4.16}$$

Alternately from (5.4.11),

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x), \tag{5.4.17}$$

$${}_{n|}\ddot{a}^{(m)}_{x} = \alpha(m)\,{}_{n|}\ddot{a}_{x} - \beta(m)\,{}_nE_x. \tag{5.4.18}$$

Backward recursion formulas can be developed directly for m-thly life annuities, and the reader is asked to do this in Exercise 5.16 for an m-thly annuity-due. A more direct approach, however, would be to use the recursions of Section 5.3 and then adjust from annual to m-thly life annuities by means of (5.4.11), (5.4.17), and (5.4.18) or the equivalent formulas (5.4.14), (5.4.15), and (5.4.16).

The distribution of the present value of the payments of a life annuity-immediate with m-thly payments can be explored in steps analogous with those for the annuity-due. For example, the present-value random variable, Y, would be

$$Y = a^{(m)}_{\overline{K+(J/m)}|} = \frac{1 - v^{K+(J/m)}}{i^{(m)}} \tag{5.4.19}$$

for the whole life annuity-immediate with m-thly payments. This exploration leads to the following formula analogous to (5.4.5):

$$1 = i\,a_x + (1+i)\,A_x = i^{(m)}\,a^{(m)}_x + \left(1 + \frac{i^{(m)}}{m}\right)A^{(m)}_x. \tag{5.4.20}$$

The meaning here is that an investment of 1 will produce interest at the end of each interest period plus the repayment of the unit together with interest then due at the end of the interest period in which death occurs.

The actuarial present values for the annuities-immediate can also be obtained by adjusting the actuarial present values of the corresponding life annuities-due. For instance,

$$a_{\overline{n}|}^{(m)} = \ddot{a}_{\overline{n}|}^{(m)} - \frac{1}{m},$$

$$a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m}(1 - {}_nE_x). \tag{5.4.21}$$

5.5 Apportionable Annuities-Due and Complete Annuities-Immediate

With discrete annuities each payment is made either for the following period (an annuity-due) or for the preceding period (an annuity-immediate). A question may arise about having an adjustment for the payment period of death. For instance, suppose that a life insurance contract is purchased by annual payments payable at the beginning of each contract year. If the insured dies 1 month after making an annual payment, a refund of premium for the 11 months the insured did not complete in the policy year of death might seem appropriate. As another example, if a retirement income life annuity-immediate provides annual payments and the annuitant dies 1 month before the due date of the next payment, there might be a final payment for the 11-month period that the annuitant has survived since the last full payment. Consider first the appropriate size for the adjustment payment in such cases.

Let us examine the first case above. The insured dies at time T after paying a full yearly premium of 1 at time K. Assume that the premium is earned or accrued at a uniform rate over the year following the payment. In this case the rate of accrual, c, is given by $c\,\bar{a}_{\overline{1}|} = 1$. If accrual ceases at death, then $c\,\bar{s}_{\overline{T-K}|}$ has been earned to that date, while

$$(1+i)^{T-K} - c\,\bar{s}_{\overline{T-K}|} = 1 \times (1+i)^{T-K} - \frac{\bar{s}_{\overline{T-K}|}}{\bar{a}_{\overline{1}|}} = \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

is unearned and is the amount to be refunded. The present-value random variable, at time 0, of all the payments less the refund is

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+1}|} - v^T \frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}} \\ &= \ddot{a}_{\overline{K+1}|} - \frac{v^T - v^{K+1}}{d} \\ &= \frac{1 - v^T}{d} = \ddot{a}_{\overline{T}|}. \end{aligned} \tag{5.5.1}$$

When the annual rate of payments is 1, the actuarial present value at time 0 of the payments is denoted by $\ddot{a}_x^{(1)}$:

$$\ddot{a}_x^{(1)} = \mathrm{E}[\ddot{a}_{\overline{T}|}] = \mathrm{E}\left[\frac{\delta}{d}\,\bar{a}_{\overline{T}|}\right] = \frac{\delta}{d}\,\bar{a}_x. \tag{5.5.2}$$

This type of life annuity-due, one with a refund for the period between the time of death and the end of the period represented by the last full regular payment, is called an ***apportionable annuity-due.***

We can extend this idea to annuities that are paid more frequently than once a year. As in Section 5.4, we define $J = \lfloor (T - K)m \rfloor$ to be the number of m-ths of a year completed in the year of death, so $K + (J + 1)/m - T$ is the length of the period to be compensated for by the refund. The accrual rate is given by $c\,\bar{a}_{\overline{1/m}|} = 1/m$. Then, proceeding as above,

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+(J+1)/m}|} - v^T \left(\frac{\bar{a}_{\overline{K+(J+1)/m-T}|}}{m\bar{a}_{\overline{1/m}|}} \right) \\ &= \ddot{a}_{\overline{K+(J+1)/m}|} - \frac{v^T - v^{K+(J+1)/m}}{d^{(m)}} \\ &= \frac{1 - v^T}{d^{(m)}} = \ddot{a}_{\overline{T}|}^{(m)}. \end{aligned} \tag{5.5.3}$$

When the annual rate of payments is 1, the actuarial present value of payments, less the refund, is $\ddot{a}_x^{\{m\}}$:

$$\ddot{a}_x^{\{m\}} = \mathrm{E}\left[\frac{1 - v^T}{d^{(m)}}\right] = \frac{\delta}{d^{(m)}}\,\bar{a}_x. \tag{5.5.4}$$

Alternatively, using the second line of (5.5.3),

$$\begin{aligned} \ddot{a}_x^{\{m\}} &= \mathrm{E}\left[\frac{1 - v^{K+(J+1)/m}}{d^{(m)}}\right] - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] \\ &= \ddot{a}_x^{(m)} - \mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right]. \end{aligned} \tag{5.5.5}$$

The second term on the right-hand side of (5.5.5) is the actuarial present value of the refund. Using the ideas developed in Exercise 4.19 we have

$$\mathrm{E}\left[\frac{v^T - v^{K+(J+1)/m}}{d^{(m)}}\right] = \frac{\bar{A}_x - A_x^{(m)}}{d^{(m)}}. \tag{5.5.6}$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$\frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x$$

and

$$\ddot{a}_x^{\{m\}} = \ddot{a}_x^{(m)} - \frac{i}{d^{(m)}}\left(\frac{1}{\delta} - \frac{1}{i^{(m)}}\right) A_x. \tag{5.5.7}$$

Let us now develop a parallel theory for annuities-immediate. Assume the annuitant dies at time T after receiving a last regular payment of size $1/m$ at time $K + J/m$, where $J = \lfloor (T - K)m \rfloor$. Now $T - K - (J/m)$ is the length of the period to be compensated for by an additional payment. Assume that each payment is accrued at a uniform rate over the m-th of the year preceding its payment. In this case the rate of accrual, c, is given by $c\,\bar{s}_{\overline{1/m}|} = 1/m$. If accrual ceases at death, an appropriate payment at death is that portion of the next payment that has been accrued to date and is given by $c\,\bar{s}_{\overline{T-K-(J/m)}|} = \bar{s}_{\overline{T-K-(J/m)}|} / (m\bar{s}_{\overline{1/m}|})$. The present value, at time 0, of all of the payments is

$$\begin{aligned} Y &= a^{(m)}_{\overline{K+J/m}|} + v^T \left(\frac{\bar{s}_{\overline{T-K-(J/m)}|}}{m\bar{s}_{\overline{1/m}|}} \right) \\ &= a^{(m)}_{\overline{K+J/m}|} + \frac{v^{K+J/m} - v^T}{i^{(m)}} \\ &= \frac{1 - v^T}{i^{(m)}} = a^{(m)}_{\overline{T}|}. \end{aligned} \tag{5.5.8}$$

When the annual rate of payments is 1, the actuarial present value at 0 of the payments is denoted by $\mathring{a}^{(m)}_x$. When $m = 1$, the (1) in the notation is omitted for this annuity:

$$\mathring{a}^{(m)}_x = \mathrm{E}\left[\frac{1 - v^T}{i^{(m)}} \right] = \frac{\delta}{i^{(m)}}\,\bar{a}_x. \tag{5.5.9}$$

Alternatively, using the second line of (5.5.8),

$$\begin{aligned} \mathring{a}^{(m)}_x &= \mathrm{E}[a^{(m)}_{\overline{K+J/m}|}] + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}} \right] \\ &= a^{(m)}_x + \mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}} \right]. \end{aligned} \tag{5.5.10}$$

The second term on the right-hand side of (5.5.10) is the actuarial present value of the final partial payment. Using the ideas developed in Exercise 4.19, we have

$$\mathrm{E}\left[\frac{v^{K+J/m} - v^T}{i^{(m)}} \right] = \frac{(1 + i)^{1/m} A^{(m)}_x - \bar{A}_x}{i^{(m)}}. \tag{5.5.11}$$

Under the uniform distribution of death assumption for each year of age, this becomes

$$\frac{i}{i^{(m)}} \left(\frac{1}{d^{(m)}} - \frac{1}{\delta} \right) A_x$$

and

$$\mathring{a}^{(m)}_x = a^{(m)}_x + \frac{i}{i^{(m)}} \left(\frac{1}{d^{(m)}} - \frac{1}{\delta} \right) A_x. \tag{5.5.12}$$

This type of life annuity-immediate, one with a partial payment for the period between the last full payment and the time of death, is called a ***complete annuity-immediate.***

For use in subsequent material, (5.5.3) and (5.5.8) seem to be most useful. We illustrate this in the following example.

Example 5.5.1

Compare the variances of the present-value random variables for the complete annuity-immediate and apportionable annuity-due.

Solution:

For the apportionable annuity-due, we have

$$\begin{aligned} \mathrm{Var}(\ddot{a}_{\overline{T}|}^{(m)}) &= \mathrm{Var}\left(\frac{1 - v^T}{d^{(m)}}\right) && \text{from (5.5.3)} \\ &= \frac{\mathrm{Var}(v^T)}{(d^{(m)})^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(d^{(m)})^2}. \end{aligned}$$

For the complete annuity-immediate, we have

$$\begin{aligned} \mathrm{Var}(a_{\overline{T}|}^{(m)}) &= \mathrm{Var}\left(\frac{1 - v^T}{i^{(m)}}\right) && \text{from (5.5.8)} \\ &= \frac{\mathrm{Var}(v^T)}{(i^{(m)})^2} \\ &= \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(i^{(m)})^2}. \end{aligned}$$

Since $i^{(m)}$ is larger than $d^{(m)}$, and in fact $i^{(m)} = d^{(m)}\,(1 + i)^{1/m}$, the variance of the complete annuity-immediate is the smaller. ▼

5.6 Notes and References

Taylor (1952) presents new formulas analogous to (5.4.6). Various inquiries into the probability distribution of annuity costs are made by Boermeester (1956), Fretwell and Hickman (1964), and Bowers (1967). This work is summarized by McCrory (1984). Mereu (1962) gives a means of calculating annuity values directly from Makeham constants. The use of the floor function, $\lfloor t \rfloor$, in actuarial science, in particular with respect to actuarial present values of life annuities, is found in Shiu (1982) and in the discussions to that paper. Complete and apportionable annuities are involved, explicitly or implicitly, in papers by Rasor and Greville (1952), Lauer (1967), and Scher (1974) and in the discussions thereto.

Exercises

Section 5.2

5.1. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\bar{a}_{20}$, $\bar{a}_{50}$, $\bar{a}_{80}$

b. $\text{Var}(\bar{a}_{\overline{T}|})$ for $x = 20, 50, 80$.
[Hint: Use (5.2.8), (5.2.9), and (4.4.4).]

5.2. Using the values obtained in Exercise 5.1, calculate the standard deviation and the coefficient of variation, σ / μ, of the following present-value random variables.

a. Individual annuities issued at ages 20, 50, 80 with life incomes of 1,000 per year payable continuously.

b. A group of 100 annuities, each issued at age 50, with life income of 1,000 per year payable continuously.

5.3. Show that $\text{Var}(\bar{a}_{\overline{T}|})$ can be expressed as

$$\frac{2}{\delta}(\bar{a}_x - {}^2\bar{a}_x) - \bar{a}_x^2,$$

where ${}^2\bar{a}_x$ is based on the force of interest 2δ.

5.4. Calculate $\text{Cov}(\delta\, \bar{a}_{\overline{T}|}, v^T)$.

5.5. If a deterministic (rate function) approach is adopted, (5.2.27) could be taken as the starting point for the development of a theory of continuous life annuities. For this, we would begin with

$$\frac{d\bar{a}_y}{dy} = [\mu(y) + \delta]\bar{a}_y - 1 \qquad x \le y < \omega,$$

$$\bar{a}_y = 0 \qquad \omega \le y.$$

a. Use the integrating factor $\exp\{-\int_0^y [\mu(z) + \delta]dz\}$ to solve the differential equation to obtain (5.2.3).

b. Use the integrating factor $e^{-\delta y}$ to obtain

$$\bar{a}_x = \bar{a}_{\overline{\omega - x}|} - \int_x^\omega e^{-\delta(y-x)}\, \bar{a}_y\, \mu(y)\, dy$$

and give an interpretation of it in words.

5.6. Assume that $\mu(x + t) = \mu$ and the force of interest is δ for all $t \ge 0$.

a. If $Y = \bar{a}_{\overline{T}|}$, $0 \le T$, display the formula for the distribution function of Y.

b. If

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & 0 \le T < n \\ \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

c. If

$$Y = \begin{cases} 0 & 0 \le T < n \\ \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

d. If

$$Y = \begin{cases} \bar{a}_{\overline{n}|} & 0 \le T < n \\ \bar{a}_{\overline{T}|} & T \ge n, \end{cases}$$

display the formula for the distribution function of Y.

5.7. By considering the integral $\int_0^{n+1} v^t \, {}_tp_x \, dt$ and breaking it in two different ways into subintervals (first, from 0 to 1 and 1 to $n + 1$ and then 0 to n and n to $n + 1$), establish a backward recursion formula for the n-year temporary life annuity based on a fixed n-year temporary period. What starting value is appropriate for this recursion formula?

5.8. By considering the integral $\int_n^{\infty} v^t \, {}_tp_x \, dt$ and breaking it into subintervals from n to $n + 1$ and $n + 1$ to ∞, establish a backward recursion formula for the n-year deferred whole life annuity based on a fixed n-year deferral period. What starting value is appropriate for this recursion formula?

5.9. Combine the result from Exercise 5.7 with the first line of (5.2.25) to establish a backward recursion formula for $u(x) = \bar{a}_{x:\overline{n}|}$. What starting value is appropriate for this recursion formula?

5.10. If the probabilities come from an aggregate table, establish (5.2.6) by a probabilistic derivation starting with a rewrite of (5.2.3) as

$$\bar{a}_x = \mathrm{E}[\bar{a}_{\overline{T}|}] = \mathrm{E}[\bar{a}_{\overline{T}|} \,|0 \le T < 1] \Pr(0 \le T < 1) + \mathrm{E}[\bar{a}_{\overline{T}} \,|1 \le T] \Pr(1 \le T).$$

Section 5.3

5.11. Show that

$$\mathrm{Var}(a_{\overline{K}|}) = \mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \frac{\mathrm{Var}(v^{K+1})}{d^2}.$$

5.12. Prove and interpret the given relations:

a. $a_{x:\overline{n}|} = {}_1E_x \, \ddot{a}_{x+1:\overline{n}|}$

b. ${}_{n|}a_x = \dfrac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x$.

5.13. Using (5.3.13), prove and interpret the following relation in words:

$$A_{x:\overline{n}|} = v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}.$$

5.14. Obtain an alternative expression for the variance given in Example 5.3.1 by starting with

$$Y^2 = \begin{cases} \dfrac{1 - 2v^K + v^{2K}}{i^2} = \dfrac{2(1 - v^K) - (1 - v^{2K})}{i^2} & K = 0, 1, n - 1 \\ (a_{\overline{n}|})^2 & K = n, n + 1 \ldots . \end{cases}$$

Section 5.4

5.15. Assume a uniform distribution of deaths over each year of age. Simplify

$$\sum_{k=0}^{\infty} {}_kp_x\, v^{k+1} \left[\sum_{j=0}^{m-1} ({}_{j/m}p_{x+k}\ {}_{1/m}q_{x+k+j/m})\ \ddot{s}^{(m)}_{\overline{1-(j+1)/m}|} \right]$$

for use in interpretation of (5.4.8).

5.16. Consider an m-thly temporary life annuity-due that pays 1 per annum to an annuitant age x for $y - x$ years.

a. Express the current payment form of the actuarial present value of the above annuity as a sum of that for payments in the first year and for the remaining $y - x - 1$ years.

b. Express the actuarial present value of payments in the first year in terms of $\alpha(m)$ and $\beta(m)$ under an assumption of the uniform distribution of deaths within each year of age.

c. Find the form of the $c(x)$ and $d(x)$ expressions for a recursion relation for such an annuity and indicate a starting value.

5.17. Using (5.4.10), derive alternative formulas to (5.4.17) and (5.4.18).

5.18. Show that the annuity-immediate analogue for (5.4.6) is

$$a^{(m)}_x = s^{(m)}_{\overline{1}|}\, a_x + \frac{1}{i^{(m)}} \left[(1 + i)A_x - \left(1 + \frac{i^{(m)}}{m}\right) A^{(m)}_x \right],$$

and that under the assumption of a uniform distribution of deaths in each year of age, this becomes

$$a^{(m)}_x = s^{(m)}_{\overline{1}|}\, a_x + (1 + i)\,\frac{1 - \ddot{a}^{(m)}_{\overline{1}|}}{i^{(m)}}\, A_x.$$

5.19. Show that the annuity-immediate analogues for (5.4.7) are

$$a^{(m)}_x = \frac{1 - (1 + i^{(m)}/m)A^{(m)}_x}{i^{(m)}} = a^{(m)}_{\overline{\infty}|} - \ddot{a}^{(m)}_{\overline{\infty}|}\, A^{(m)}_x$$

and that under the assumption of a uniform distribution of deaths in each year of age these become

$$a_x^{(m)} = \alpha(m)a_x + \frac{1 - \ddot{a}_{\overline{1}|}^{(m)}}{i^{(m)}}.$$

5.20. a. Use (5.4.3) as a starting point to verify that

$$\lim_{m\to\infty} \ddot{a}_x^{(m)} = \bar{a}_x.$$

b. Use (5.4.10) and the result in (a) to show

$$\bar{a}_x \cong a_x + \frac{1}{2}.$$

5.21. Using the traditional approximation given in (5.4.10), establish the following:

a. $a_x^{(m)} \cong a_x + \dfrac{m-1}{2m}$

b. $a_{x:\overline{n}|}^{(m)} \cong a_{x:\overline{n}|} + \dfrac{m-1}{2m}(1 - {}_nE_x)$

c. ${}_{n|}a_x^{(m)} \cong {}_{n|}a_x + \dfrac{m-1}{2m}\,{}_nE_x.$

5.22. a. Develop a formula for $\ddot{s}_{25:\overline{40}|}^{(m)}$ in terms of $\ddot{s}_{25:\overline{40}|}$.

b. On the basis of the Illustrative Life Table with interest at the effective annual rate of 6%, calculate the values of

(i) $\ddot{a}_{25:\overline{40}|}^{(12)}$ (ii) $\ddot{s}_{25:\overline{40}|}^{(12)}$.

5.23. The actuarial present value of a standard increasing temporary life annuity with respect to (x) with

- Yearly income of 1 in the first year, 2 in the second year, and so on, ending with n in the n-th year,
- Payments made m-thly on a due basis

is denoted by $(I\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_{x:\overline{n-k}|}^{(m)}.$$

5.24. The actuarial present value of a standard decreasing temporary life annuity with respect to (x) with

- Yearly income of n in the first year, $n-1$ in the second year, and so on, ending with 1 in the n-th year,
- Payments, made m-thly on a due basis

is denoted by $(D\ddot{a})_{x:\overline{n}|}^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=1}^{n} \ddot{a}_{x:\overline{k}|}^{(m)}.$$

5.25. If in Exercise 5.23 the yearly income does not cease at age $x + n$ but continues at the level n while (x) survives therafter, the actuarial present value is denoted by $(I_{\overline{n}|}\,\ddot{a})_x^{(m)}$.

a. Display the present-value random variable, Y, for this annuity as a function of the K and J random variables.

b. Show that the actuarial present value can be expressed as

$$\sum_{k=0}^{n-1} {}_{k|}\ddot{a}_x^{(m)}.$$

5.26. Verify the formula

$$\delta(\bar{I}\bar{a})_{\overline{T}|} + T\,v^T = \bar{a}_{\overline{T}|},$$

where T represents the future lifetime of (x). Use it to prove that

$$\delta(\bar{I}\bar{a})_x + (\bar{I}\bar{A})_x = \bar{a}_x,$$

where $(\bar{I}\bar{a})_x$ is the actuarial present value of a life annuity to (x) under which payments are being made continuously at the rate of t per annum at time t.

5.27. From $a_{x:\overline{n}|}^{(m)} = a_x^{(m)} + a_{\overline{n}|}^{(m)} - a_{x:\overline{n}|}^{(m)}$, show that the assumption of uniform distribution of deaths in each year of age leads to

$$a_{x:\overline{n}|}^{(m)} = \frac{i}{i^{(m)}}\left[a_{\overline{n}|} + v^n\,{}_np_x\,a_{x+n} + \left(\frac{1}{d} - \frac{1}{d^{(m)}}\right) v^n\,{}_np_x\,A_{x+n}\right].$$

Section 5.5

5.28. Establish and interpret the following formulas:

a. $1 = i^{(m)}\,\mathring{a}_x^{(m)} + \bar{A}_x$

b. $1 = d^{(m)}\,\ddot{a}_x^{(m)} + \bar{A}_x$

c. $\mathring{a}_{x:\overline{n}|}^{(m)} = (\delta / i^{(m)})\,\bar{a}_{x:\overline{n}|}$

d. $\ddot{a}_{x:\overline{n}|}^{(m)} = (\delta / d^{(m)})\,\bar{a}_{x:\overline{n}|}$

e. $\ddot{a}_{x:\overline{n}|}^{(m)} = (1 + i)^{1/m}\,\mathring{a}_{x:\overline{n}|}^{(m)}$.

5.29. Let $H(m) = \ddot{a}_x^{(m)} - \mathring{a}_x^{(m)}$. Prove that $H(m) \geq 0$ and $\lim_{m\to\infty} H(m) = 0$.

Miscellaneous

5.30. For $0 \leq t \leq 1$ and the assumption of a uniform distribution of deaths in each year of age, show that

a. $\ddot{a}_{x+t} = \dfrac{(1 + it)\ddot{a}_x - t(1 + i)}{1 - t\,q_x}$

b. ${}_t\ddot{a}_x = v^t[(1 + it)\ddot{a}_x - t(1 + i)]$

c. ${}_{1}\,{}_{t|}\ddot{a}_{x+t} = \dfrac{(1+i)^t}{1-tq_x}\,(\ddot{a}_x - 1)$

d. $A_{x+t} = \dfrac{1+it}{1-tq_x}\,A_x - \dfrac{t\,q_x}{1-t\,q_x}$.

5.31. Obtain formulas for the evaluation of a life annuity-due to (x) with an initial payment of 1 and with annual payments increasing thereafter by
a. 3% of the initial annual payment
b. 3% of the previous year's annual payment.

5.32. Express $(\bar{D}\bar{a})_{x:\overline{n}|}$ as an integral and prove the formula

$$\frac{\partial}{\partial n}\,(\bar{D}\bar{a})_{x:\overline{n}|} = \bar{a}_{x:\overline{n}|}.$$

5.33. Give an expression for the actuarial accumulated value at age 70 of an annuity with the following monthly payments:
- 100 at the end of each month from age 30 to 40
- 200 at the end of each month from age 40 to 50
- 500 at the end of each month from age 50 to 60
- 1,000 at the end of each month from age 60 to 70.

5.34. Derive a simplified expression for the actuarial present value for a 25-year term insurance payable immediately on the death of (35), under which the death benefit in case of death at age $35 + t$ is $\bar{s}_{\overline{t}|}$, $0 \le t \le 25$. Interpret your result.

5.35. Derive a simplified expression for the actuarial present value for an n-year term insurance payable at the end of the year of death of (x), under which the death benefit in case of death in year $k+1$ is $\ddot{s}_{\overline{k+1}|}$, $0 \le k < n$. Interpret your result.

5.36. Obtain a simplified expression for

$$(I\ddot{a})^{(12)}_{x:\overline{25}|} - (Ia)^{(12)}_{x:\overline{25}|}.$$

5.37. Consider an n-year deferred continuous life annuity of 1 per year as an insurance with probability of claim, ${}_{n}p_x$, and random amount of claim, $v^n\,\bar{a}_{\overline{T}|}$. Here T has p.d.f., ${}_{t}p_{x+n}\,\mu_x(n+t)$. Apply (2.2.13) to show that the variance of the insurance equals

$$v^{2n}\,{}_{n}p_x\,(1 - {}_{n}p_x)\,\bar{a}^2_{x+n} + v^{2n}\,{}_{n}p_x\,\frac{{}^2\bar{A}_{x+n} - (\bar{A}_{x+n})^2}{\delta^2}$$

and verify that this reduces to (5.2.21).

5.38. Write the discrete analogue of the variance formula in Exercise 5.37.

5.39. Consider the indicator random variable, I_k, defined by

$$I_k = \begin{cases} 1 & T(x) \geq k \\ 0 & T(x) < k. \end{cases}$$

Show the following:

a. The present value of a life annuity to (x), with annual payment b_k on survival to age $x + k$, $k = 0, 1, 2, \ldots$, can be written as

$$\sum_{k=0}^{\infty} v^k \, b_k \, I_k.$$

b. $\mathrm{E}[I_j I_k] = {}_kp_x \qquad j \leq k$

$\mathrm{Cov}(I_j, I_k) = {}_kp_x \, {}_jq_x \qquad j \leq k.$

c. $\mathrm{Var}\left(\sum_{k=0}^{\infty} v^k \, b_k \, I_k\right) = \sum_{k=0}^{\infty} v^{2k} \, b_k^2 \, {}_kp_x \, {}_kq_x + 2 \sum_{k=0}^{\infty} \sum_{j<k} v^{j+k} \, b_j \, b_k \, {}_kp_x \, {}_jq_x.$

5.40. If a left superscript 2 indicates that interest is at force 2δ, show that

a. ${}^2A_x = 1 - (2d - d^2) \, {}^2\ddot{a}_x$

b. $\mathrm{Var}(v^{K+1}) = 2d(\ddot{a}_x - {}^2\ddot{a}_x) - d^2 \, (\ddot{a}_x^2 - {}^2\ddot{a}_x)$

c. $\mathrm{Var}(\ddot{a}_{\overline{K+1}|}) = \dfrac{2}{d} \, (\ddot{a}_x - {}^2\ddot{a}_x) - (\ddot{a}_x^2 - {}^2\ddot{a}_x).$

5.41. a. Expand, in terms of powers of δ, the annuity coefficients $\alpha(m)$ and $\beta(m)$.

b. What do the expansions in (a) become for $m = \infty$?

5.42. Use Jensen's inequality to show, for $\delta > 0$, that

a. $\bar{a}_x < \bar{a}_{\overline{e_x}|}$ b. $a_x < a_{\overline{e_x}|} \qquad x < \omega - 1.$

5.43. If $g(x)$ is a non-negative function and X is a random variable with p.d.f. $f(x)$, justify the inequality

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \, f(x) \, dx \geq k \, \Pr[g(X) \geq k] \qquad k > 0$$

and use it to show that

$$\bar{a}_x \geq \bar{a}_{\overline{e_x}|} \Pr(\bar{a}_{\overline{T}|} \geq \bar{a}_{\overline{e_x}|}) = \bar{a}_{\overline{e_x}|} \Pr(T \geq \mathring{e}_x).$$

5.44. A unit is to be used to purchase a combination benefit consisting of a life income of I per year payable continuously while (x) survives and an insurance of J payable immediately on the death of (x). Write the present-value random variable for this combination and give its mean and variance.

5.45. Using the assumption of a uniform distribution of deaths in each year of age and the Illustrative Life Table with interest at the effective annual rate of 6%, calculate

a. $\ddot{a}_{40}^{(12)}$ b. $\ddot{a}_{40:\overline{30}|}^{(12)}$ c. ${}_{30|}\ddot{a}_{40}^{(12)}.$

5.46. If $A''_{x:\overline{m}|}$ and $\ddot{a}''_{x:\overline{m}|}$ are actuarial present values calculated using

- An interest rate of i for the first n years, $n < m$, and
- An interest rate i' for the remaining $m - n$ years,

show algebraically and interpret

a. $A''_{x:\overline{m}|} = 1 - d\,\ddot{a}_{x:\overline{n}|} - v^n\,{}_np_x\,d'\,\ddot{a}'_{x+n:\overline{m-n}|}$

b. $A''_{x:\overline{m}|} = 1 - d'\,\ddot{a}''_{x:\overline{m}|} + (d' - d)\,\ddot{a}_{x:\overline{n}|}$.

5.47. Show that

$$\frac{d\ddot{a}_x}{di} = -v(Ia)_x,$$

where

$$(Ia)_x = \sum_{t=1}^{\infty} t\,v^t\,{}_tp_x,$$

and interpret the relation.

5.48. Show that a constant increase in the force of mortality has the same effect on $\ddot{a}_x$ as a constant increase in the force of interest, but that this is not the case for $\ddot{a}_x^{(m)}$ evaluated by $\alpha(m)\,\ddot{a}_x - \beta(m)$.

5.49. Show that

$$\alpha(m) - \beta(m)d = \ddot{a}_{\overline{1}|}^{(m)}.$$

5.50. Show that, if $q_x < (i^{(2)}/2)^2$, the approximation

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x),$$

in the special case with $n = 1$, $m = 2$, leads to

$$\ddot{a}_{x:\overline{1}|}^{(2)} > \ddot{a}_{\overline{1}|}^{(2)}.$$

5.51. Consider the following portfolio of annuities-due currently being paid from the assets of a pension fund.

Age	Number of Annuitants
65	30
75	20
85	10

Each annuity has an annual payment of 1 as long as the annuitant survives. Assume an earned interest rate of 6% and a mortality as given in the Illustrative Life Table. For the present value of these obligations of the pension fund, calculate

a. The expectation

b. The variance

c. The 95th percentile of its distribution.
For parts (b) and (c), assume the lives are mutually independent.

Computing Exercises

5.52. a. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a life annuity-due of 1 per annum for ages 13 to 140.
b. Compare your values to those given in Table 2A.

5.53. For your Illustrative Life Table with $i = 0.06$, calculate the actuarial present value of a temporary life annuity-due of 1 per annum payable to age 65 for ages 13 to 64.

5.54. Using your Illustrative Life Table with the assumption of a uniform distribution of deaths within each year of age and $i = 0.06$, calculate the actuarial present value of a 10-year temporary life annuity-due of 1 per annum issued at ages 13 to 99 using the results of Exercise 5.7.

5.55. a. Add $\alpha(m)$ and $\beta(m)$ to the interest functions calculated and stored in your Illustrative Life Table. Refer to Exercise 4.31.
b. Determine $\alpha(12)$ and $\beta(12)$ at $i = 0.06$ and compare your results to those given in Example 5.4.1.
[Remark: We suggest using the series derived in Exercise 5.41 for accurate results with small interest rates.]

5.56. Using your Illustrative Life Table with $i = 0.06$ and the assumption of uniform distribution of deaths over each year of age, calculate the actuarial present value of a temporary life annuity of 1 per annum payable continuously to age 65 for ages 13 to 64.

5.57. Let Y be the present-value random variable for a continuous 10-year temporary life annuity of 1 per annum commencing at age 60. On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.08$, calculate the mean and variance of Y.

5.58. Let Y be the present-value random variable for a life annuity-due of 1 per annum, payable monthly to (65). On the basis of your Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.05$, calculate the mean and variance of Y.

5.59. Use the Illustrative Life Table with uniform distribution of deaths over each year of age and $i = 0.07$ to determine $a_{30:\overline{20}|}$.



BENEFIT PREMIUMS

6.1 Introduction

In Chapters 4 and 5 we discussed actuarial present values of the payments of various life insurances and annuities. These ideas are combined in this chapter to determine the level of life annuity payments necessary to buy, or fund, the benefits of an insurance or annuity contract. In practice individual life insurance is usually purchased by a life annuity of ***contract premiums***—the insurance contract specifies the premium to be paid. Contract premiums provide for benefits, expenses of initiating and maintaining the insurance, and margins for profit and for offsetting possible unfavorable experience. The premiums studied in this chapter are determined only by the pattern of benefits and premiums and do not consider expenses, profit, or contingency margins.

In Chapter 1 we discussed the idea that determination of the insurance premium requires the adoption of a ***premium principle.*** Example 6.1.1 illustrates the application of three such premium principles. All three principles are based on the impact of the insurance on the wealth of the insuring organization. The random variable that gives the present value at issue of the insurer's loss, if the insurance is contracted at a certain premium level, is the key in the model for the principles. Principle I requires that the loss random variable be positive with no more than a specified probability. Principles II and III are based on the expected utility of the insurer's wealth as discussed in Section 1.3. We will see that Principle II, which uses a linear utility function, could also be characterized as requiring that the loss random variable have zero expected value.

Example 6.1.1

An insurer is planning to issue a policy to a life age 0 whose curtate-future-lifetime, K, is governed by the p.f.

$$ {}_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4. $$

The policy will pay 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Find the annual premium, P, as determined by:

a. Principle I: P will be the least annual premium such that the insurer has probability of a positive financial loss of at most 0.25.
b. Principle II: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = x$, will be indifferent between accepting and not accepting the risk.
c. Principle III: P will be the annual premium such that the insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between accepting and not accepting the risk.

For all three parts assume the insurer will use an annual effective interest rate of $i = 0.06$.

Solution:

For $K = k$ and an arbitrary premium, P, the present value of the financial loss at policy issue is $l(k) = v^{k+1} - P\ddot{a}_{\overline{k+1}|} = (1 + P/d)\, v^{k+1} - P/d$, $k = 0, 1, 2, 3, 4$. The corresponding loss random variable is $L = v^{K+1} - P\ddot{a}_{\overline{K+1}|}$.

a. Since $l(k)$ decreases as k increases, the requirement of Principle I will hold if P is such that $v^2 - P\,\ddot{a}_{\overline{2}|} = 0$. Then the financial loss is positive for only $K = 0$, which has probability $0.2 < 0.25$. Thus, for this principle, $P = 1/\ddot{s}_{\overline{2}|} = 0.45796$.
b. By an extension of (1.3.6), we seek the premium P such that $u(w) = \mathrm{E}[u(w - L)]$. By Principle II $u(x) = x$ so we have

$$w = \mathrm{E}[w - L] = w - \mathrm{E}[L].$$

Thus, Principle II is equivalent to requiring that P be chosen so that $\mathrm{E}[L] = 0$. For this example, we require

$$\sum_{k=0}^{4} (v^{k+1} - P\,\ddot{a}_{\overline{k+1}|})\,\Pr(K = k) = 0 \tag{6.1.1}$$

which gives $P = 0.30272$.

c. Again by (1.3.6) and now using the utility function in Principle III, we have

$$-e^{-0.1w} = \mathrm{E}[-e^{-0.1(w-L)}] = -e^{-0.1w}\,\mathrm{E}[e^{0.1L}].$$

Thus, Principle III is equivalent to requiring that P be chosen so that $\mathrm{E}[e^{0.1L}] = 1$. Here, we require

$$\sum_{k=0}^{4} \exp[0.1\,(v^{k+1} - P\,\ddot{a}_{\overline{k+1}|})]\,\Pr(K = k) = 1, \tag{6.1.2}$$

which gives $P = 0.30628$.

These three results are summarized below.

Outcome k	Probability ${}_k q_0$	General Formula	Present Value of Financial Loss When Premium Is by Principle			exp(0.1L)
			I	II	III	III
0	0.2	$v^1 - P\,\ddot{a}_{\overline{1}\rceil}$	0.48544	0.64067	0.63712	1.06579
1	0.2	$v^2 - P\,\ddot{a}_{\overline{2}\rceil}$	0	0.30169	0.29477	1.02992
2	0.2	$v^3 - P\,\ddot{a}_{\overline{3}\rceil}$	−0.45796	−0.01811	−0.02819	0.99718
3	0.2	$v^4 - P\,\ddot{a}_{\overline{4}\rceil}$	−0.89000	−0.31981	−0.33287	0.96726
4	0.2	$v^5 - P\,\ddot{a}_{\overline{5}\rceil}$	−1.29758	−0.60443	−0.62031	0.93985
	Premium		0.45796	0.30272	0.30628	
	Expected Value		−0.43202	0.00000	−0.12020	1.00000

The table shows that for this example the decision makers adopting principles I and III reduce their risk in the sense that they are demanding an expected present value of loss to be negative. ▼

Premiums defined by Principle I are known as ***percentile premiums.*** Although the principle is attractive on the surface, it is easy to show that it can lead to quite unsatisfactory premiums. Such cases are examined in Example 6.2.3.

Principle II has many applications in practice. To formalize its concepts, we define the insurer's loss, L, as the random variable of the present value of benefits to be paid by the insurer less the annuity of premiums to be paid by the insured. Principle II is called the ***equivalence principle*** and has the requirement that

$$\mathrm{E}[L] = 0. \tag{6.1.3}$$

We will speak of ***benefit premiums*** as those satisfying (6.1.3). Equivalently, benefit premiums will be such that

E[present value of benefits] = E[present value of benefit premiums].

Methods developed in Chapters 4 and 5 for calculating these actuarial present values can be used to reduce this equality to a form that can be solved for the premiums. For instance, in Example 6.1.1, which has constant benefit premiums and constant benefits of 1, equation (6.1.1) can be rewritten as $A_0 = P\,\ddot{a}_0$, and $\ddot{a}_0$ can be calculated as

$$\sum_{k=0}^{4} v^k\,{}_k p_0.$$

When the equivalence principle is used to determine a single premium at policy issue for a life insurance or a life annuity, the premium is equal to the actuarial present value of benefit payments and is called the ***single benefit premium.***

Premiums based on Principle III, using an exponential utility function, are known as ***exponential premiums.*** Exponential premiums are nonproportional in the sense that the premium for the policy with a benefit level of 10 is more than 10 times

the premium for a policy with a benefit level of 1 (see Exercise 6.2). This is consistent for a risk averse utility function.

6.2 Fully Continuous Premiums

The basic concepts involved in the determination of annual benefit premiums using the equivalence principle will be illustrated first for the case of the fully continuous level annual benefit premium for a unit whole life insurance payable immediately on the death of (x). For any continuously paid premium, $\bar{P}$, consider

$$l(t) = v^t - \bar{P}\,\bar{a}_{\overline{t}|}, \tag{6.2.1}$$

the present value of the loss to the insurer if death occurs at time t.

We note that $l(t)$ is a decreasing function of t with $l(0) = 1$ and $l(t)$ approaching $-\bar{P}/\delta$ as $t \to \infty$. If t_0 is the time when $l(t_0) = 0$, death before t_0 results in a positive loss, whereas death after t_0 produces a negative loss, that is, a gain. Figure 6.2.1 later in this section illustrates these ideas.

We now consider the loss random variable,

$$L = l(T) = v^T - \bar{P}\,\bar{a}_{\overline{T}|}, \tag{6.2.2}$$

corresponding to the loss function $l(t)$. If the insurer determines his premium by the equivalence principle, the premium is denoted by $\bar{P}(\bar{A}_x)$ and is such that

$$\mathrm{E}[L] = 0. \tag{6.2.3}$$

It follows from (4.2.6) and (5.2.3) that

$$\bar{A}_x - \bar{P}(\bar{A}_x)\bar{a}_x = 0,$$

or

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}. \tag{6.2.4}$$

Remark:
In this chapter we continue to suppress the select notation except in situations in which it is necessary or helpful to eliminate ambiguity.

The variance of L can be used as a measure of the variability of losses on an individual whole life insurance due to the random nature of time-until-death. When $\mathrm{E}[L] = 0$,

$$\mathrm{Var}(L) = \mathrm{E}[L^2]. \tag{6.2.5}$$

For the loss in (6.2.2), we have

$$\mathrm{Var}(v^T - \bar{P}\,\bar{a}_{\overline{T}|}) = \mathrm{Var}\left[v^T - \frac{\bar{P}(1 - v^T)}{\delta}\right]$$

$$= \mathrm{Var}\left[v^T\left(1 + \frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]$$

$$= \text{Var}\left[v^T\left(1 + \frac{\bar{P}}{\delta}\right)\right]$$

$$= \text{Var}\,(v^T)\left(1 + \frac{\bar{P}}{\delta}\right)^2$$

$$= [{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1 + \frac{\bar{P}}{\delta}\right)^2. \qquad (6.2.6)$$

For the premium determined by the equivalence principle, we can use (6.2.4) and (5.2.8), $\delta\bar{a}_x + \bar{A}_x = 1$, to rewrite (6.2.6) as

$$\text{Var}(L) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2}. \qquad (6.2.7)$$

Example 6.2.1

Calculate $\bar{P}(\bar{A}_x)$ and Var(L) with the assumptions that the force of mortality is a constant $\mu = 0.04$ and the force of interest $\delta = 0.06$.

Solution:

These assumptions yield $\bar{a}_x = 10$, $\bar{A}_x = 0.4$, and ${}^2\bar{A}_x = 0.25$. Using (6.2.4), we obtain

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = 0.04,$$

and from (6.2.7)

$$\text{Var}(L) = \frac{0.25 - 0.16}{(0.6)^2} = 0.25. \qquad \blacktriangledown$$

By reference to (6.2.6) we can see that the numerator of this last expression can be interpreted as the variance of the loss, $v^T - \bar{A}_x$, associated with a single-premium whole life insurance. This latter variance is 0.09, and hence the standard deviation of the loss associated with this annual premium insurance is $\sqrt{0.25/0.09} = 5/3$ times the standard deviation of the loss in the single-premium case. Additional uncertainty about the present value of the premium income increases the variability of losses due to the random nature of time-until-death.

In Example 6.2.1, $\bar{P}(\bar{A}_x) = 0.04$, the constant force of mortality. We can confirm this as a general result by using parts of Examples 4.2.3 and 5.2.1. Under the constant force of mortality assumption,

$$\bar{A}_x = \frac{\mu}{\mu + \delta}$$

and

$$\bar{a}_x = \frac{1}{\mu + \delta},$$

thus

$$\bar{P}(\bar{A}_x) = \frac{\mu(\mu + \delta)^{-1}}{(\mu + \delta)^{-1}} = \mu,$$

which does not depend on the force of interest or the age at issue.

Using the equivalence principle, as in (6.1.3), we can determine formulas for annual premiums of a variety of fully continuous life insurances. Our general loss is

$$b_T v_T - \bar{P}\, Y = Z - \bar{P}\, Y \tag{6.2.8}$$

where

- b_t and v_t are, respectively, the benefit amount and discount factor defined in connection with (4.2.1)
- $\bar{P}$ is a general symbol for a fully continuous net annual premium
- Y is a continuous annuity random variable as defined, for example, in (5.2.13), and
- Z is defined by (4.2.2).

Application of the equivalence principle yields

$$\mathrm{E}[b_T v_T - \bar{P}\, Y] = 0$$

or

$$\bar{P} = \frac{\mathrm{E}[b_T v_T]}{\mathrm{E}[Y]}.$$

These ideas are used to display annual premium formulas in Table 6.2.1.

It is of interest to note how these steps can be used for an n-year deferred whole life annuity of 1 per year payable continuously. In this case $b_T v_T = 0$, $T \le n$ and $b_T v_T = \bar{a}_{\overline{T-n}|} v^n$, $T > n$. Then,

$$\begin{aligned}\mathrm{E}[b_T v_T] &= {}_n p_x\, \mathrm{E}[\bar{a}_{\overline{T-n}|} v^n | T > n] \\ &= v^n\, {}_n p_x\, \bar{a}_{x+n} = A^{\,1}_{x:\overline{n}|}\, \bar{a}_{x+n}.\end{aligned}$$

In practice, however, deferred life annuities usually provide some type of death benefit during the period of deferment. One contract of this type is examined in Example 6.6.2.

Example 6.2.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.2.1).

Fully Continuous Benefit Premiums

Plan	Loss Components: $b_T v_T$	Loss Components: $\bar{P}\ Y$ Where Y Is	Premium Formula $\bar{P} = \frac{\mathrm{E}[b_T v_T]}{\mathrm{E}[Y]}$
Whole life insurance	$1\ v^T$	$\bar{a}_{\overline{T}\rceil}$	$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}$
n-Year term insurance	$1\ v^T$ 0	$\bar{a}_{\overline{T}\rceil}$, $T \le n$ $\bar{a}_{\overline{n}\rceil}$, $T > n$	$\bar{P}(\bar{A}^1_{x:\overline{n}\rceil}) = \frac{\bar{A}^1_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year endowment insurance	$1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil}$, $T \le n$ $\bar{a}_{\overline{n}\rceil}$, $T > n$	$\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \frac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
h-Payment* whole life insurance	$1\ v^T$ $1\ v^T$	$\bar{a}_{\overline{T}\rceil}$, $T \le h$ $\bar{a}_{\overline{h}\rceil}$, $T > h$	${}_h\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\overline{h}\rceil}}$
h-Payment,* n-year endowment insurance	$1\ v^T$ $1\ v^T$ $1\ v^n$	$\bar{a}_{\overline{T}\rceil}$, $T \le h$ $\bar{a}_{\overline{h}\rceil}$, $h < T \le n$ $\bar{a}_{\overline{h}\rceil}$, $T > n$	${}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \frac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{h}\rceil}}$
n-Year pure endowment	0 $1\ v^n$	$\bar{a}_{\overline{T}\rceil}$, $T \le n$ $\bar{a}_{\overline{n}\rceil}$, $T > n$	$\bar{P}(A^1_{x:\overline{n}\rceil}) = \frac{A^1_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
n-Year† deferred whole life annuity	0 $\bar{a}_{\overline{T-n}\rceil}\ v^n$	$\bar{a}_{\overline{T}\rceil}$, $T \le n$ $\bar{a}_{\overline{n}\rceil}$, $T > n$	$\bar{P}({}_{n\rceil}\bar{a}_x) = \frac{A^1_{x:\overline{n}\rceil}\ \bar{a}_{x+n}}{\bar{a}_{x:\overline{n}\rceil}}$

*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.

†The annuity product described above provides no death benefits and has a level premium with premiums payable for n years. A different, perhaps more realistic, design for an n-year level premium-deferred annuity is given in Example 6.6.2.

Solution:

Using the notation of (4.2.11), we have

$$\mathrm{Var}(L) = \mathrm{Var}\left\{Z_3\left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right] - \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right\}.$$

We now use (4.2.10) to obtain

$$\mathrm{Var}(L) = \left[1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta}\right]^2 [{}^2\bar{A}_{x:\overline{n}\rceil} - (\bar{A}_{x:\overline{n}\rceil})^2].$$

Formula (5.2.14) can be rewritten as

$$(\delta\,\bar{a}_{x:\overline{n}\rceil})^{-1} = 1 + \frac{\bar{P}(\bar{A}_{x:\overline{n}\rceil})}{\delta},$$

which implies that

$$\text{Var}(L) = \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{(\delta\,\bar{a}_{x:\overline{n}|})^2}. \qquad \blacktriangledown$$

The two identities, (5.2.8) and (5.2.15), can be used to derive relationships among continuous benefit premiums. For example, starting with (5.2.8),

$$\begin{aligned}
\delta\,\bar{a}_x + \bar{A}_x &= 1, \\
\delta + \bar{P}(\bar{A}_x) &= \frac{1}{\bar{a}_x}, \\
\bar{P}(\bar{A}_x) &= \frac{1}{\bar{a}_x} - \delta \\
&= \frac{1 - \delta\,\bar{a}_x}{\bar{a}_x} \\
&= \frac{\delta\,\bar{A}_x}{1 - \bar{A}_x}.
\end{aligned} \tag{6.2.9}$$

Starting with (5.2.14) we obtain

$$\begin{aligned}
\delta\,\bar{a}_{x:\overline{n}|} + \bar{A}_{x:\overline{n}|} &= 1, \\
\delta + \bar{P}(\bar{A}_{x:\overline{n}|}) &= \frac{1}{\bar{a}_{x:\overline{n}|}}, \\
\bar{P}(\bar{A}_{x:\overline{n}|}) &= \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta \\
&= \frac{1 - \delta\,\bar{a}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} \\
&= \frac{\delta\,\bar{A}_{x:\overline{n}|}}{1 - \bar{A}_{x:\overline{n}|}}.
\end{aligned} \tag{6.2.10}$$

Verbal interpretations of the discrete analogues of (6.2.9) and (6.2.10) are given in Example 6.3.4.

The premiums discussed so far in this section are benefit premiums, those derived from the equivalence principle. We now turn to an example that describes two ways that percentile premiums give unsatisfactory results.

Example 6.2.3

Find the 25th percentile premium for an insured age 55 for the following plans of insurance:

a. 20-year endowment
b. 20-year term
c. 10-year term.

Assume a fully continuous basis with a force of interest, $\delta = 0.06$ and mortality following the Illustrative Life Table.

Solution:

a. The loss function for 20-year endowment insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20$$

$$= v^{20} - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20$$

and is a nonincreasing function of T. Thus the values of T for which the loss L is to be positive, which are to have probability of 0.25, are those values below $\xi_T^{0.25}$. Since $l_{55} = 86{,}408.60$ and $l_{70.617} = 64{,}806.45$ (by linear interpolation), $\Pr(T < 15.617) = 0.25$. Thus, the premium required by the 25th percentile principle is that which sets the loss at $T = 15.617$ equal to zero and is $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$.

b. The loss function for 20-year term insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 20$$

$$= \quad - \bar{P}\bar{a}_{\overline{20}|} \qquad T \geq 20.$$

This is still a nonincreasing function of T, and since $\Pr(T < 15.617) = 0.25$, the premium required by the 25th percentile principle is again $v^{15.617}/\bar{a}_{\overline{15.617}|} = 0.03865$. It is, of course, unsatisfactory that the same premium is generated for two different plans of insurance, particularly since the benefit premium at this age for 20-year endowment is almost two times that for 20-year term.

c. The loss function for a 10-year term insurance is

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T < 10$$

$$= - \bar{P}\bar{a}_{\overline{10}|} \qquad T \geq 10.$$

If the premium is set at zero, then $\Pr(L > 0) = \Pr(T < 10)$, and this probability equals, by the Illustrative Life Table,

$$\frac{(l_{55} - l_{65})}{l_{55}} = 0.1281.$$

Thus, zero is the least non-negative annual premium such that the insurer's probability of financial loss is at most 0.25. In this case $\Pr(L > 0) = 0.1281$, and $\bar{P} = 0$ the 25th percentile premium. The benefit premium in this case is 70% of that for 20-year term insurance. ▼

The conclusion from this example is that the percentile premium principle yields conflicting results for insurances on a single individual. Its use will be minimal in what follows.

For a whole life insurance, as defined in the first row of Table 6.2.1,

$$L = v^T - \bar{P}\bar{a}_{\overline{T}|} \qquad T \geq 0.$$

The d.f. of L can be developed as follows:

$$\begin{aligned} F_L(u) &= \Pr(L \le u) \\ &= \Pr\left[v^T - \bar{P}\left(\frac{1 - v^T}{\delta}\right) \le u \right] \\ &= \Pr\left(v^T \le \frac{\delta u + \bar{P}}{\delta + \bar{P}} \right) \\ &= \Pr\left[T \ge -\frac{1}{\delta} \log\left(\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right) \right] \\ &= 1 - F_T\left(-\frac{1}{\delta} \log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right] \right) \qquad -\frac{\bar{P}}{\delta} < u. \end{aligned} \tag{6.2.11}$$

The p.d.f. of L is

$$\frac{d}{du} F_L(u) = f_L(u) = f_T\left(-\frac{1}{\delta} \log\left[\frac{\delta u + \bar{P}}{\delta + \bar{P}}\right] \right)\left(\frac{1}{\delta u + \bar{P}}\right) \qquad -\frac{\bar{P}}{\delta} < u. \tag{6.2.12}$$

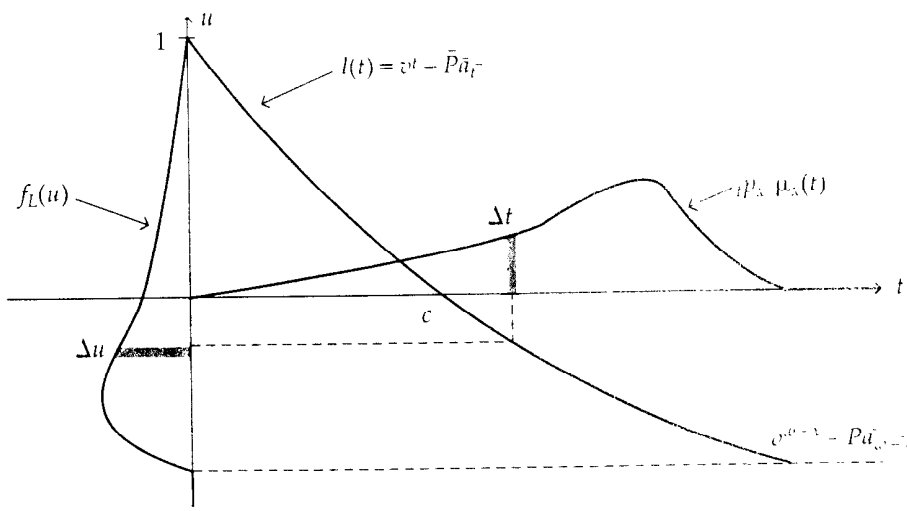
Using the language of decision analysis, we can say that the determination of the premium $\bar{P}$ is equivalent to selecting the distribution of L, given by (6.2.11), that is optimal from the viewpoint of the premium principle adopted by the decision maker. This principle reflects the preferences of the decision maker.

Schematic diagrams of $l(t)$, the p.d.f. of $T(x)$, and the induced p.d.f. of L are combined in Figure 6.2.1.

The set of d.f.'s of L is indexed by the parameter $\bar{P}$. The value of $\bar{P}$ is selected by the premium principle adopted. For illustration, use Figure 6.2.1 where $\Pr(T \le c) = \Pr(L > 0)$ and this probability is taken as 0.25. We assume that the

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Schematic Diagrams of $l(t)$ and the p.d.f.'s of $T(x)$ and L



value of $\bar{P}$ will be obtained by solving $F_L(0) = 1 - 0.25 = 0.75$. This illustration uses a percentile premium principle with the probability for a positive value of L set at 0.25.

It is evident from Figure 6.2.1 that the events $(T \leq c)$ and $(L > 0)$ are equivalent in the sense that the occurrence of one of the two events implies the occurrence of the other. To continue our illustration, if the decision maker has adopted the percentile premium principle with $\Pr(L > 0) = p$, then $\Pr(T \leq c) = p$, where $c = \xi_T^p$, the $100p$-th percentile of the distribution of T. Furthermore, because of the equivalence of these two events, the premium can be determined from an equation involving the loss function, that is, from

$$v^{\xi_T^p} - \bar{P}\bar{a}_{\overline{\xi_T^p}|} = 0,$$

or

$$\bar{P} = \frac{v^{\xi_T^p}}{\bar{a}_{\overline{\xi_T^p}|}} = \frac{1}{\bar{s}_{\overline{\xi_T^p}|}}. \tag{6.2.13}$$

Because $\bar{P}$ is the rate of payment into a fund that will provide a unit payment at time ξ_T^p, there is intuitive support for the result. The accumulation $\bar{s}_{\overline{1}|}/\bar{s}_{\overline{\xi_T^p}|}$ will be less than 1 with probability p and greater than 1 with probability $1 - p$.

Example 6.2.4

This example builds on Example 6.2.3, except that $T(55)$ has a De Moivre distribution, with p.d.f.

$${}_tp_{55}\,\mu_{55}(t) = 1/45 \qquad 0 < t < 45.$$

For the three loss variables, display the d.f. of L and determine the parameter $\bar{P}$ as the smallest non-negative number such that $\Pr(L > 0) \leq 0.25$.

Solution:

a. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = v^{20} - \bar{P}\bar{a}_{\overline{20}|}$ induced by the constraint on L if $T \geq 20$, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned} F_L(u) &= 0 && u \leq v^{20} - \bar{P}\bar{a}_{\overline{20}|} \\ &= 1 + \frac{1}{0.06}\,\frac{\log\left[(0.06u + \bar{P})/(0.06 + \bar{P})\right]}{45} && v^{20} - \bar{P}\bar{a}_{\overline{20}|} < u \leq 1 \\ &= 1 && 1 < u. \end{aligned}$$

Figure 6.2.2a is a diagram of the d.f. associated with a 20-year endowment insurance. This figure provides a graphical way of thinking of premium determination using the percentile principle. The d.f. within the set of d.f.'s indexed by $\bar{P}$ that crosses the vertical axis at 0.75 is sought. Analytically this means that the premium is determined by solving, for $\bar{P}$,

$$F_L(0) = 0.75$$

$$= 1 + \frac{1}{0.06}\,\frac{\log[\bar{P}/(0.06 + \bar{P})]}{45} = 0.75,$$

or

$$\log\left(\frac{\bar{P}}{0.06 + \bar{P}}\right) = -0.675,$$

and

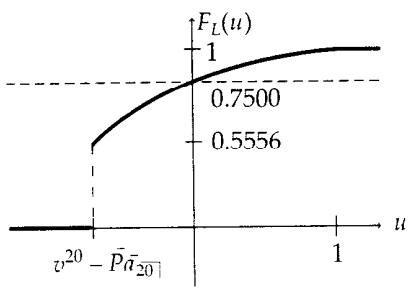
$$\bar{P} = \frac{0.06e^{-0.675}}{(1 - e^{-0.675})}$$

$$= \frac{1}{\bar{s}_{\overline{11.25}|}}$$

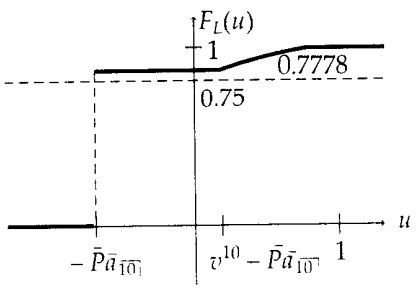
$$= 0.06224.$$

In view of the discussion about (6.2.11), this is not surprising. The 25th percentile of the De Moivre distribution of T in this example is $\xi_T^{0.25} = 11.25$.

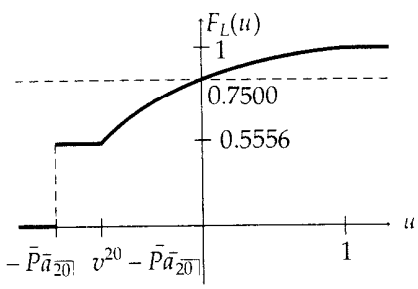
Distribution Functions of L Developed in Example 6.2.4



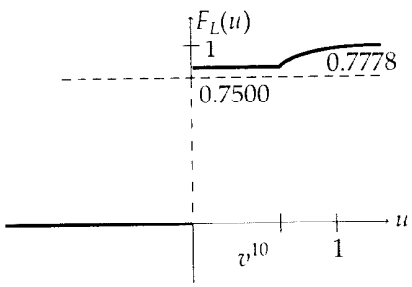
(a) d.f. of L, 20-year endowment insurance $\bar{P} = 0.06244$



(c) d.f. of 10-year term insurance $\bar{P} = 0.06244$



(b) d.f. of L, 20-year term insurance $\bar{P} = 0.06244$



(d) d.f. of 10-year term insurance $\bar{P} = 0$

For comparison, the benefit, or equivalence principle, premium is

$$\bar{P}(\bar{A}_{55:\overline{20}|}) = \frac{\int_0^{20} (v^t/45)\,dt + (25/45)\,v^{20}}{\int_0^{20} v^t\,[1 - (t/45)]\,dt} = 0.04456.$$

b. Adapting (6.2.11) with recognition of the jump in the d.f. of L at $u = -\bar{P}\bar{a}_{\overline{20}|}$, induced by the constraint on the 20-year term insurance loss variable, we have the following set of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned}
F_L(u) &= 0 && u \le -\bar{P}\bar{a}_{\overline{20}|} \\
&= \frac{25}{45} && -\bar{P}\bar{a}_{\overline{20}|} < u \le v^{20} - \bar{P}\bar{a}_{\overline{20}|} \\
&= 1 + \frac{1}{0.06}\,\frac{\log\,[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} && v^{20} - \bar{P}\bar{a}_{\overline{20}|} < u \le 1 \\
&= 1 && 1 < u.
\end{aligned}$$

A diagram of the d.f. associated with a 20-year term insurance is shown in Figure 6.2.2b. The premium is determined by solving $F_L(0) = 0.75$ for $\bar{P}$. Using part (a) we find, once more, that $\bar{P} = 0.06224$.

c. Adapting (6.2.11), with recognition of the jump in the d.f. at $u = -\bar{P}\bar{a}_{\overline{10}|}$ induced by the constraint on L for 10-year term insurance, we have the family of d.f.'s indexed by $\bar{P}$:

$$\begin{aligned}
F_L(u) &= 0 && u \le -\bar{P}\bar{a}_{\overline{10}|} \\
&= \frac{35}{45} && -\bar{P}\bar{a}_{\overline{10}|} < u \le v^{10} - \bar{P}\bar{a}_{\overline{10}|} \\
&= 1 + \frac{1}{0.06}\,\frac{\log\,[(0.06u + \bar{P})/(0.06 + \bar{P})]}{45} && v^{10} - \bar{P}\bar{a}_{\overline{10}|} < u \le 1 \\
&= 1 && 1 < u.
\end{aligned}$$

It is tempting to conjecture that $\bar{P} = 0.06224$, as it was in parts (a) and (b), when we observe that the only nonconstant values of the d.f. have the same formula as in the earlier parts of this example. When we observe that for any u in the interval $(-\bar{P}\bar{a}_{\overline{10}|},\ v^{10} - \bar{P}\bar{a}_{\overline{10}|})$, $F_L(u) = 35/45 > 0.75$, it appears that the conjecture is wrong. Figure 6.2.2c displays the d.f. of L when $\bar{P} = 0.06224$ and confirms this judgment. As in Example 6.2.3, try $\bar{P} = 0$. The corresponding d.f. of L is

$$\begin{aligned}
F_L(u) &= 0 && u \le 0 \\
&= \frac{35}{45} && 0 < u \le v^{10} \\
&= 1 + \frac{1}{0.06}\,\frac{\log u}{45} && v^{10} < u \le 1 \\
&= 1 && 1 < u
\end{aligned}$$

and the probability of a positive value of L is

$$\Pr(L > 0) = \frac{10}{45} < 0.25.$$

This is illustrated in Figure 6.2.2d.

As in Example 6.2.3c, the specifications for applying the percentile premium principle leads to $\bar{P} = 0$, an anomalous result from a business perspective. ▼

6.3 Fully Discrete Premiums

In Section 6.2 we have discussed the theory of fully continuous benefit premiums. In this section we consider annual premium insurances like the one that appeared in Example 6.1.1; that is, the sum insured is payable at the end of the policy year in which death occurs, and the first premium is payable when the insurance is issued. Subsequent premiums are payable on anniversaries of the policy issue date while the insured survives during the contractual premium payment period. The set of annual premiums form a life annuity-due. This model does not conform to practice but is of historic importance in the development of actuarial theory.

Under these circumstances, the level annual benefit premium for a unit whole life insurance is denoted by P_x, where the absence of $(\bar{A}_x)$ means that the insurance is payable at the end of the policy year of death. The loss for this insurance is

$$L = v^{K+1} - P_x \, \ddot{a}_{\overline{K+1}|} \qquad K = 0, 1, 2, \ldots . \tag{6.3.1}$$

The equivalence principle requires that $\mathrm{E}[L] = 0$, or

$$\mathrm{E}[v^{K+1}] - P_x \, \mathrm{E}[\ddot{a}_{\overline{K+1}|}] = 0,$$

which yields

$$P_x = \frac{A_x}{\ddot{a}_x}. \tag{6.3.2}$$

This is the discrete analogue of (6.2.4).

Using (5.3.7) in place of (5.2.8) in steps parallel to those taken in obtaining (6.2.7), we obtain

$$\mathrm{Var}(L) = \frac{{}^{2}A_x - (A_x)^2}{(d\,\ddot{a}_x)^2}. \tag{6.3.3}$$

Example 6.3.1

If

$$ {}_{k|}q_x = c(0.96)^{k+1} \qquad k = 0, 1, 2, \ldots $$

where $c = 0.04/0.96$ and $i = 0.06$, calculate P_x and Var(L).

Solution:

First we exhibit the components of (6.3.2),

$$ A_x = c \sum_{k=0}^{\infty} (1.06)^{-k-1} (0.96)^{k+1} = 0.40, $$

$$ \ddot{a}_x = \frac{1 - A_x}{d} = 10.60. $$

Then using (6.3.2) we obtain

$$ P_x = \frac{A_x}{\ddot{a}_x} = 0.0377. $$

For Var(L), we calculate

$$ {}^2A_x = c \sum_{k=0}^{\infty} [(1.06)^2]^{-k-1} (0.96)^{k+1} = 0.2445. $$

Therefore,

$$ \begin{aligned} \text{Var}(L) &= \frac{0.2445 - 0.1600}{[(0.06)(10.60)/(1.06)]^2} \\ &= 0.2347. \end{aligned} $$

▼

There is a connection between Examples 6.2.1 and 6.3.1. Since

$$ {}_{k|}q_x = \int_k^{k+1} {}_tp_x \, \mu_x(t) \, dt \qquad k = 0, 1, 2, \ldots \tag{6.3.4} $$

for the situation described in Example 6.3.1, we have

$$ \frac{0.04}{0.96} (0.96)^{k+1} = \int_k^{k+1} {}_tp_x \, \mu_x(t) \, dt. $$

If the force of mortality is a constant, μ, it follows that

$$ \frac{0.04}{0.96} (0.96)^{k+1} = e^{-(k+1)\mu} (e^{\mu} - 1), $$

and then $e^{-\mu} = 0.96$ and $\mu = 0.0408$. The geometric distribution, with p.f.

$$ {}_{k|}q_x = \frac{0.04}{0.96} (0.96)^{k+1}, $$

is a discrete version of the exponential distribution with $\mu = 0.0408$. Formula (6.3.4) provides the bridge between the discrete and continuous versions. The fully continuous annual benefit premium corresponding to $P_x = 0.0377$ in Example 6.3.1 would be $\bar{P}(\bar{A}_x) = \mu = 0.0408$.

Continuing to use the equivalence principle, we can determine formulas for annual benefit premiums for a variety of fully discrete life insurances. Our general loss will be

$$b_{K+1}v_{K+1} - PY$$

where

- b_{k+1} and v_{k+1} are, respectively, the benefit and discount functions defined in (4.3.1)
- P is a general symbol for an annual premium paid at the beginning of each policy year during the premium paying period while the insured survives and
- Y is a discrete annuity random variable as defined, for example, in connection with (5.3.9).

Application of the equivalence principle yields

$$E[b_{K+1}v_{K+1} - PY] = 0,$$

or

$$P = \frac{E[b_{K+1}v_{K+1}]}{E[Y]}.$$

These ideas are used in Table 6.3.1 to display premium formulas for fully discrete insurances.

Example 6.3.2

Express the variance of the loss, L, associated with an n-year endowment insurance, in terms of actuarial present values (see the third row of Table 6.3.1).

Solution:

We start with the notation of Table 6.3.1. Let

$$Z = \begin{cases} v^{K+1} & K = 0, 1, \ldots, n-1 \\ v^n & K = n, n+1, \ldots \end{cases}$$

Then we can write, by reference to the third row of Table 6.3.1,

$$L = Z - P_{x:\overline{n}|}\frac{1-Z}{d};$$

therefore we have

$$\text{Var}(L) = \text{Var}\left[Z\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d}\right].$$

We can use the rule of moments to find Var(Z), as indicated in Table 4.3.1, and then obtain

Fully Discrete Annual Benefit Premiums

Plan	Loss Components: $b_{K+1}v_{K+1}$	Loss Components: $P\,Y$ Where Y Is	Premium Formula $P = \frac{E[b_{K+1}v_{K+1}]}{E[Y]}$
Whole life insurance	$1\,v^{K+1}$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, 2, \ldots$	$P_x = \frac{A_x}{\ddot{a}_x}$
n-Year term insurance	$1\,v^{K+1}$ 0	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rceil}$, $K = n, n+1, \ldots$	$P^1_{x:\overline{n}\rceil} = \frac{A^1_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}}$
n-Year endowment insurance	$1\,v^{K+1}$ $1\,v^n$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rceil}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\rceil} = \frac{A_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{n}\rceil}}$
h-Payment whole life insurance	$1\,v^{K+1}$ $1\,v^{K+1}$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rceil}$, $K = h, h+1, \ldots$	${}_hP_x = \frac{A_x}{\ddot{a}_{x:\overline{h}\rceil}}$
h-Payment, n-year endowment insurance	$1\,v^{K+1}$ $1\,v^{K+1}$ $1\,v^n$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, h-1$ $\ddot{a}_{\overline{h}\rceil}$, $K = h, \ldots, n-1$ $\ddot{a}_{\overline{h}\rceil}$, $K = n, n+1, \ldots$	${}_hP_{x:\overline{n}\rceil} = \frac{A_{x:\overline{n}\rceil}}{\ddot{a}_{x:\overline{h}\rceil}}$
n-Year pure endowment	0 $1\,v^n$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rceil}$, $K = n, n+1, \ldots$	$P_{x:\overline{n}\rceil}^{\frac{1}{}} = \frac{A_{x:\overline{n}\rceil}^{\frac{1}{}}}{\ddot{a}_{x:\overline{n}\rceil}}$
n-Year deferred whole life annuity	0 $\ddot{a}_{\overline{K+1-n}\rceil}v^n$	$\ddot{a}_{\overline{K+1}\rceil}$, $K = 0, 1, \ldots, n-1$ $\ddot{a}_{\overline{n}\rceil}$, $K = n, n+1, \ldots$	$P({}_{n\rceil}\ddot{a}_x) = \frac{A_{x:\overline{n}\rceil}^{\frac{1}{}}\,\ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}\rceil}}$

$$\text{Var}(L) = \left(1 + \frac{P_{x:\overline{n}\rceil}}{d}\right)^2 [{}^2A_{x:\overline{n}\rceil} - (A_{x:\overline{n}\rceil})^2].$$

Formula (5.3.13) and the entry from the third row of Table 6.3.1 can be combined as follows:

$$d\,\ddot{a}_{x:\overline{n}\rceil} + A_{x:\overline{n}\rceil} = 1,$$

$$1 + \frac{P_{x:\overline{n}\rceil}}{d} = \frac{1}{d\,\ddot{a}_{x:\overline{n}\rceil}}.$$

Therefore, the variance we seek is

$$\frac{{}^2A_{x:\overline{n}\rceil} - (A_{x:\overline{n}\rceil})^2}{(d\,\ddot{a}_{x:\overline{n}\rceil})^2}. \tag{6.3.5}$$

▼

Example 6.3.3

Consider a 10,000 fully discrete whole life insurance. Let π denote an annual premium for this policy and $L(\pi)$ denote the loss-at-issue random variable for one such policy on the basis of the Illustrative Life Table, 6% interest and issue age 35.

a. Determine the premium, π_a, such that the distribution of $L(\pi_a)$ has mean 0. Calculate the variance of $L(\pi_a)$.
b. Approximate the smallest non-negative premium, π_b, such that the probability is less than 0.5 that the loss $L(\pi_b)$ is positive. Find the variance of $L(\pi_b)$.
c. Determine the premium, π_c, such that the probability of a positive total loss on 100 such independent policies is 0.05 by the normal approximation.

Solution:

a. By the equivalence principle, (6.1.3),

$$\pi_a = 10{,}000\, P_{35} = 10{,}000\, \frac{A_{35}}{\ddot{a}_{35}}$$

$$= \frac{1287.194}{15.39262}$$

$$= 83.62.$$

From (6.3.3)

$$\text{Var}[L(\pi_a)] = (10{,}000)^2\, \frac{{}^2A_{35} - (A_{35})^2}{(d\,\ddot{a}_{35})^2}$$

$$= 10^8\, \frac{0.0348843 - (0.1287194)^2}{[(0.06/1.06)(15.39262)]^2}$$

$$= \frac{1{,}831{,}562}{0.7591295}$$

$$= 2{,}412{,}713.$$

b. We want π_b such that

$$\Pr[L(\pi_b) > 0] < 0.5,$$

or in terms of curtate-future-lifetime, K,

$$\Pr(10{,}000v^{K+1} - \pi_b\, \ddot{a}_{\overline{K+1}|} > 0) < 0.5.$$

From the Illustrative Life Table, ${}_{42}p_{35} = 0.5125101$ and ${}_{43}p_{35} = 0.4808964$. Therefore, if π_b is chosen so that

$$10{,}000v^{43} - \pi_b\, \ddot{a}_{\overline{43}|} = 0,$$

then $\Pr[L(\pi_b) > 0)] = \Pr(K < 42) < 0.5$. Thus,

$$\pi_b = \frac{10{,}000}{\ddot{s}_{\overline{43}|}} = 50.31.$$

Using the fully discrete analogue of (6.2.6) we can write

$$\begin{aligned}\text{Var}[L(\pi_b)] &= (10{,}000)^2[{}^2A_{35} - (A_{35})^2]\left(1 + \frac{\pi_b}{10{,}000}\frac{1}{d}\right)^2 \\ &= (1{,}831{,}562)(1.18567) \\ &= 2{,}171{,}630.\end{aligned}$$

c. With a premium π_c, the loss on one policy is

$$L(\pi_c) = 10{,}000v^{K+1} - \pi_c\ddot{a}_{\overline{K+1}|} = \left(10{,}000 + \frac{\pi_c}{d}\right)v^{K+1} - \frac{\pi_c}{d},$$

and its expectation and variance are as follows:

$$\begin{aligned}\text{E}[L(\pi_c)] &= \left(10{,}000 + \frac{\pi_c}{d}\right)A_{35} - \frac{\pi_c}{d} \\ &= (0.1287194)\left(10{,}000 + \frac{\pi_c}{d}\right) - \frac{\pi_c}{d}\end{aligned}$$

and

$$\begin{aligned}\text{Var}[L(\pi_c)] &= \left(10{,}000 + \frac{\pi_c}{d}\right)^2[{}^2A_{35} - (A_{35})^2] \\ &= \left(10{,}000 + \frac{\pi_c}{d}\right)^2(0.01831562).\end{aligned}$$

For each of 100 such policies each loss $L_i(\pi_c)$ is distributed like $L(\pi_c)$, $i = 1, 2, \ldots, 100$ and

$$S = \sum_{i=1}^{100} L_i(\pi_c)$$

for the total loss on the portfolio. Then

$$\text{E}[S] = 100\ \text{E}[L(\pi_c)],$$

and, using the assumption of independent policies,

$$\text{Var}(S) = 100\ \text{Var}[L(\pi_c)].$$

To determine π_c so that $\Pr(S > 0) = 0.05$ by the normal approximation, we want

$$\frac{0 - \text{E}[S]}{\sqrt{\text{Var}(S)}} = 1.645,$$

$$10\left\{\frac{-\text{E}[L(\pi_c)]}{\sqrt{\text{Var}[L(\pi_c)]}}\right\} = 1.645,$$

$$10\left[\frac{-A_{35}[10{,}000 + (\pi_c/d)] + (\pi_c/d)}{[10{,}000 + (\pi_c/d)]\sqrt{{}^2A_{35} - (A_{35})^2}}\right] = 1.645.$$

Thus

$$\pi_c = 10{,}000\, d \left[\frac{(0.1645)\sqrt{{}^2A_{35} - (A_{35})^2} + A_{35}}{1 - (A_{35} + 0.1645\sqrt{{}^2A_{35} - (A_{35})^2})} \right]$$

$$= 100.66. \qquad \blacktriangledown$$

The two identities, (5.3.7) and (5.3.13), can be used to derive relationships among discrete premiums. For example, starting with (5.3.7), we have for whole life insurances

$$d\,\ddot{a}_x + A_x = 1,$$

$$d + P_x = \frac{1}{\ddot{a}_x},$$

$$P_x = \frac{1}{\ddot{a}_x} - d$$

$$= \frac{1 - d\,\ddot{a}_x}{\ddot{a}_x}$$

$$= \frac{d\,A_x}{1 - A_x}. \tag{6.3.6}$$

Starting with (5.3.13) we obtain a similar chain of equalities for n-year endowment insurances:

$$d\,\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|} = 1,$$

$$d + P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}},$$

$$P_{x:\overline{n}|} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d$$

$$= \frac{1 - d\,\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$$

$$= \frac{d\,A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}. \tag{6.3.7}$$

Example 6.3.4

Give interpretations in words of the following equations from the (6.3.6) set:

$$\frac{1}{\ddot{a}_x} = P_x + d \tag{6.3.8}$$

and

$$P_x = \frac{d\,A_x}{1 - A_x}. \tag{6.3.9}$$

Solution:

We will use the word equivalent to mean equal in terms of actuarial present value. For (6.3.8), first note that a unit now is equivalent to a life annuity of $\ddot{a}_x^{-1}$ payable at the beginning of each year while (x) survives. A unit now is also equivalent to interest-in-advance of d at the beginning of each year while (x) survives with the repayment of the unit at the end of the year of (x)'s death; that is, $1 = \ddot{a}_x/\ddot{a}_x = d\ddot{a}_x + A_x$. The repayment of the unit at the end of the year of death is, in turn, equivalent to a life annuity-due of P_x while (x) survives. Therefore, the unit now is equivalent to $P_x + d$ at the beginning of each year during the lifetime of (x). Then $\ddot{a}_x^{-1} = P_x + d$, for each side of the equality, represents the annual payment of a life annuity produced by a unit available now.

For (6.3.9), we consider an insured (x) who borrows the single benefit premium A_x for the purchase of a single-premium unit whole life insurance. The insured agrees to pay interest-in-advance in the amount of $d\,A_x$ on the loan at the beginning of each year during survival and to repay the A_x from the unit death benefit at the end of the year of death. In essence, the insured is paying an annual benefit premium of $d\,A_x$ for an insurance of amount $1 - A_x$. Then for a full unit of insurance, the annual benefit premium must be $d\,A_x/(1 - A_x)$. ▼

Similar interpretations exist for corresponding relationships involving endowment insurances as given in the second and fifth equalities in the (6.3.7) set. There is an analogy between (6.3.8), the corresponding formula involving endowment insurances,

$$\ddot{a}_{x:\overline{n}|}^{-1} = P_{x:\overline{n}|} + d,$$

and the interest-only formula

$$\ddot{a}_{\overline{n}|}^{-1} = \ddot{s}_{\overline{n}|}^{-1} + d.$$

Example 6.3.5

Prove and interpret the formula

$$P_{x:\overline{n}|} = {}_nP_x + P_{x:\overline{n}|}^{\frac{1}{}}(1 - A_{x+n}). \tag{6.3.10}$$

Solution:

The proof is completed using entries from Table 6.3.1:

$$P_{x:\overline{n}|}\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|} = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{\frac{1}{}},$$

$${}_nP_x\ddot{a}_{x:\overline{n}|}^{\frac{1}{}} = A_x = A_{x:\overline{n}|}^{1} + A_{x:\overline{n}|}^{\frac{1}{}}\,A_{x+n}.$$

By subtraction,

$$(P_{x:\overline{n}|} - {}_nP_x)\ddot{a}_{x:\overline{n}|} = A_{x:\overline{n}|}^{\frac{1}{}}(1 - A_{x+n}),$$

from which (6.3.10) follows.

The interpretation is that both $P_{x:\overline{n}|}$ and ${}_nP_x$ are payable during the survival of (x) to a maximum of n years. During these years, both insurances provide a death benefit of 1 payable at the end of the year of the death of (x). If (x) survives the n years, $P_{x:\overline{n}|}$ provides a maturity benefit of 1, while ${}_nP_x$ provides whole life insurance without further premiums, that is, an insurance with an actuarial present value of A_{x+n}. Hence, the difference $P_{x:\overline{n}|} - {}_nP_x$ is the level annual premium for a pure endowment of $1 - A_{x+n}$. ▼

In practice, life insurances are payable soon after death rather than at the end of the policy year of death, so there is a need for annual payment, semicontinuous benefit premiums. Such premiums, following the same order used in Tables 6.2.1 and 6.3.1, are denoted by $P(\bar{A}_x)$, $P(\bar{A}^1_{x:\overline{n}|})$, $P(\bar{A}_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$. There is no need for a semicontinuous annual premium n-year pure endowment since no death benefit is involved. The equivalence principle can be applied to produce formulas like those in Table 6.3.1, but with the general symbol A replaced by $\bar{A}$. For example,

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x}. \tag{6.3.11}$$

We observe that the notation for this premium is not $\bar{P}_x$, the annual premium payable continuously for a unit whole life insurance benefit payable at the end of the year of death and equal to $A_x/\bar{a}_x$. If a uniform distribution of deaths is assumed over each year of age, we can use the notations of Section 4.4 to write

$$P(\bar{A}_x) = \frac{i}{\delta}\frac{A_x}{\ddot{a}_x} = \frac{i}{\delta}P_x,$$

$$P(\bar{A}^1_{x:\overline{n}|}) = \frac{i}{\delta}P^1_{x:\overline{n}|},$$

and

$$P(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}P^1_{x:\overline{n}|} + P_{x:\overline{n}|}^{\frac{1}{}}. \tag{6.3.12}$$

6.4 True *m*-thly Payment Premiums

If premiums are payable m times a policy year, rather than annually, with no adjustment in the death benefit, the resulting premiums are called ***true fractional premiums.*** Thus $P_x^{(m)}$ denotes the ***true level annual benefit premium,*** payable in m-thly installments at the beginning of each m-thly period, for a unit whole life insurance payable at the end of the year of death. The symbol $P^{(m)}(\bar{A}_x)$ would have the same interpretation except that the insurance is payable at the moment of death. Typically, m is 2, 4, or 12.

The development in this section stresses the payment of insurance benefits at the end of the policy year of death. Table 6.4.1 specifies the symbols and formulas for true fractional premiums for common life insurances. The premium formulas can be obtained by applying the equivalence principle.

True Fractional Benefit Premiums*

Plan	Payment of Proceeds	
	At End of Policy Year	At Moment of Death
Whole life insurance	$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}}$	$P^{(m)}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x^{(m)}}$
n-Year term insurance	$P_{x:\overline{n}|}^{1\,(m)} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}$	$P^{(m)}(\bar{A}^1_{x:\overline{n}|}) = \frac{\bar{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}$
n-Year endowment insurance	$P_{x:\overline{n}|}^{(m)} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}$	$P^{(m)}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}^{(m)}}$
h-Payment years, whole life insurance	${}_hP_x^{(m)} = \frac{A_x}{\ddot{a}_{x:\overline{h}|}^{(m)}}$	${}_hP^{(m)}(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_{x:\overline{h}|}^{(m)}}$
h-Payment years, n-year endowment insurance	${}_hP_{x:\overline{n}|}^{(m)} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}^{(m)}}$	${}_hP^{(m)}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}^{(m)}}$

*The actual amount of each fractional premium, payable m times each policy year, during the premium paying period and the survival of (x), is $P^{(m)}/m$. Note that here h refers to the number of payment years, not to the number of payments.

In some applications it is useful to write the m-thly payment premium as a multiple of the annual premium. This will be illustrated for ${}_hP_{x:\overline{n}|}^{(m)}$, the premium for a rather general insurance. The resulting formula can be modified to produce premium formulas for other common insurances. From the last row of Table 6.4.1 we have

$$ {}_hP_{x:\overline{n}|}^{(m)} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}^{(m)}}. \tag{6.4.1} $$

Since

$$ A_{x:\overline{n}|} = {}_hP_{x:\overline{n}|}\,\ddot{a}_{x:\overline{h}|}, $$

(6.4.1) can be rearranged as

$$ {}_hP_{x:\overline{n}|}^{(m)} = \frac{{}_hP_{x:\overline{n}|}\,\ddot{a}_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}^{(m)}}. \tag{6.4.2} $$

Formula (6.4.2) is used in the next chapter; it expresses the m-thly payment premium as equal to the corresponding annual payment premium times a ratio of annuity values. This ratio can be arranged in various ways each corresponding to a different formula used to express the relationship between $\ddot{a}_{x:\overline{h}|}^{(m)}$ and $\ddot{a}_{x:\overline{h}|}$ (see Exercise 6.14).

Example 6.4.1

a. Calculate the level annual benefit premium payable in semiannual installments for a 10,000, 20-year endowment insurance with proceeds paid at the end of the policy year of death (discrete) issued to (50), on the basis of the Illustrative Life Table with interest at the effective annual rate of 6%.

b. Determine the corresponding premium with proceeds paid at the moment of death (semicontinuous).

For both parts, assume a uniform distribution of deaths in each year of age.

Solution:

a. We require 10,000 $P^{(2)}_{50:\overline{20}|}$. As preliminary steps we calculate

$$d = 0.056603774,$$

$$i^{(2)} = 0.059126028,$$

$$d^{(2)} = 0.057428275,$$

$$\ddot{a}^{(2)}_{\overline{1}|} = 0.98564294,$$

$$s^{(2)}_{\overline{1}|} = 1.01478151,$$

$$\alpha(2) = s^{(2)}_{\overline{1}|}\ddot{a}^{(2)}_{\overline{1}|} = 1.0002122,$$

$$\beta(2) = \frac{s^{(2)}_{\overline{1}|} - 1}{d^{(2)}} = 0.25739081,$$

and the following actuarial present values:

$$\ddot{a}_{50:\overline{20}|} = 11.291832,$$

$$A^{1}_{50:\overline{20}|} = 0.13036536,$$

$$P^{1}_{50:\overline{20}|} = 0.01154510,$$

$${}_{20}E_{50} = 0.23047353,$$

$$A_{50:\overline{20}|} = 0.36083889,$$

$$P_{50:\overline{20}|} = 0.03195574.$$

Then, under the assumption of a uniform distribution of deaths for each year of age, the required premium can be calculated by use of (6.4.1), with $x = 50$, $n = 20$, $h = 20$, and $m = 2$. For this purpose, we calculate

$$\ddot{a}^{(2)}_{50:\overline{20}|} = \alpha(2)\ddot{a}_{50:\overline{20}|} - \beta(2)(1 - {}_{20}E_{50}) = 11.096159,$$

and then

$$10{,}000\, P^{(2)}_{50:\overline{20}|} = 325.19.$$

b. The corresponding semicontinuous premium can be obtained by multiplying the values in (a) by the ratio

$$\frac{P(\bar{A}_{50:\overline{20}|})}{P_{50:\overline{20}|}} = \frac{\bar{A}_{50:\overline{20}|}}{A_{50:\overline{20}|}}.$$

Under the uniform distribution of deaths assumption this ratio is

$$\frac{(i/\delta)\,P^{\frac{1}{}}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\frac{1}{}}}{P_{50:\overline{20}|}}, \tag{6.4.3}$$

and the result is

$$10{,}000\,P^{(2)}(\bar{A}_{50:\overline{20}|}) = 328.68. \qquad \blacktriangledown$$

6.5 Apportionable Premiums

A second type of fractional premium is the ***apportionable premium.*** Here, at death, a refund is made of a portion of the premium related to the length of time between the time of death and the time of the next scheduled premium payment. In practice this may be on a pro rata basis without interest. In this section we consider interest and view the sequence of m-thly premiums as an apportionable life annuity-due in the sense of Section 5.5. The symbols used to denote these level apportionable annual benefit premiums payable m-thly are like the symbols for true fractional premiums on the semicontinuous basis. They differ in that the superscript m is enclosed in braces rather than parentheses, for example, $P^{\{m\}}(\bar{A}_x)$. In view of the premium refund feature, it is natural to assume that the death benefit is payable at the moment of death.

Again, we use an h-payment years, n-year endowment insurance to illustrate the development of formulas for apportionable premiums paid m-thly. The equivalence principle leads to the formulas

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\,\ddot{a}^{\{m\}}_{x:\overline{h}|} = \bar{A}_{x:\overline{n}|}$$

and

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{\{m\}}_{x:\overline{h}|}}. \tag{6.5.1}$$

Utilizing the temporary annuity version of (5.5.4), we obtain

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{(\delta/d^{(m)})\bar{a}_{x:\overline{h}|}} = \frac{d^{(m)}}{\delta}\,{}_h\bar{P}(\bar{A}_{x:\overline{n}|}). \tag{6.5.2}$$

This implies that the m-thly installment is

$$\frac{1}{m}\,{}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\frac{1-v^{1/m}}{\delta} = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{1/m}|}, \tag{6.5.3}$$

and in particular, for $m = 1$,

$${}_hP^{\{1\}}(\bar{A}_{x:\overline{n}|}) = {}_h\bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{1}|}. \tag{6.5.4}$$

Formulas (6.5.3) and (6.5.4) demonstrate that these apportionable premiums are equivalent to fully continuous premiums, discounted for interest to the start of each payment period. Similar formulas exist for other types of insurance. For example, by letting h and $n \to \infty$, (6.5.4) becomes

$$P^{(1)}(\bar{A}_x) = \bar{P}(\bar{A}_x)\,\bar{a}_{\overline{1}|}. \tag{6.5.5}$$

The apportionable benefit premium $P^{(1)}(\bar{A}_x)$ and the semicontinuous benefit premium $P(\bar{A}_x)$ are both payable annually at the beginning of each year while (x) survives. Each insurance provides a unit at the death of (x). The two insurances differ only in respect to the refund provided by $P^{(1)}(\bar{A}_x)$. Thus, the difference

$$P^{(1)}(\bar{A}_x) - P(\bar{A}_x) \tag{6.5.6}$$

is a level annual benefit premium paid at the beginning of each year for the refund-of-premium feature. We verify this assertion about the expression in (6.5.6) in the following analysis.

From (5.5.1), we note that the random variable for the present value of the refund-of-premium feature is

$$\frac{P^{(1)}(\bar{A}_x)\,v^T\,\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}$$

where K and T are defined as in Chapter 3. The actuarial present value for this feature is

$$\bar{A}_x^{PR} = P^{(1)}(\bar{A}_x)\,\mathrm{E}\left[v^T\,\frac{\bar{a}_{\overline{K+1-T}|}}{\bar{a}_{\overline{1}|}}\right].$$

Using (6.5.5) we obtain

$$\begin{aligned}\bar{A}_x^{PR} &= \bar{P}(\bar{A}_x)\,\mathrm{E}\left[\frac{v^T - v^{K+1}}{\delta}\right] \\ &= \bar{P}(\bar{A}_x)\left(\frac{\bar{A}_x - A_x}{\delta}\right).\end{aligned} \tag{6.5.7}$$

The level annual benefit premium is then, by the equivalence principle,

$$P(\bar{A}_x^{PR}) = \frac{\bar{P}(\bar{A}_x)(\bar{A}_x - A_x)}{\delta\ddot{a}_x}. \tag{6.5.8}$$

Formula (6.5.7) has the following interpretation: The actuarial present value of the refund feature is the difference between the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ per year beginning at the death of (x), and the value of a continuous perpetuity of $\bar{P}(\bar{A}_x)$ payable from the end of the year of death of (x).

We return now to (6.5.6) where, by (6.5.5), we have

$$\begin{aligned} P^{(1)}(\bar{A}_x) - P(\bar{A}_x) &= \bar{P}(\bar{A}_x)\frac{d}{\delta} - \frac{\bar{A}_x}{\ddot{a}_x} \\ &= \bar{P}(\bar{A}_x)\left(\frac{d}{\delta} - \frac{\bar{a}_x}{\ddot{a}_x}\right) \\ &= \bar{P}(\bar{A}_x)\frac{d\,\ddot{a}_x - \delta\,\bar{a}_x}{\delta\,\ddot{a}_x} \\ &= \bar{P}(\bar{A}_x)\frac{\bar{A}_x - A_x}{\delta\ddot{a}_x} \\ &= P(\bar{A}_x^{PR}), \end{aligned} \tag{6.5.9}$$

as obtained in (6.5.8). This confirms our assertion about (6.5.6).

This analysis can be extended to m-thly payment premiums and to other life insurance in addition to whole life. In general,

$$P^{(m)}(\bar{A}) - P^{(m)}(\bar{A})$$

is an m-thly payment premium for the refund feature.

Example 6.5.1

If the policy of Example 6.4.1(b) is to have apportionable premiums, what increase occurs in the annual benefit premium?

Solution:

The apportionable annual benefit premium per unit of insurance is given by (6.5.2),

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \bar{P}(\bar{A}_{50:\overline{20}|})\frac{d^{(2)}}{\delta} = \frac{\bar{A}_{50:\overline{20}|}}{\bar{a}_{50:\overline{20}|}}\frac{d^{(2)}}{\delta}.$$

Under the assumption of a uniform distribution of deaths in each age interval, this becomes

$$\begin{aligned} &= \frac{(i/\delta)A^{1}_{50:\overline{20}|} + A_{50:\overline{20}|}^{\frac{1}{}}}{\alpha(\infty)\ddot{a}_{50:\overline{20}|} - \beta(\infty)(1 - {}_{20}E_{50})}\frac{d^{(2)}}{\delta} \\ &= \frac{(i/\delta)\,P^{1}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\frac{1}{}}}{\alpha(\infty) - \beta(\infty)(P^{1}_{50:\overline{20}|} + d)}\frac{d^{(2)}}{\delta}. \end{aligned}$$

Here $\alpha(\infty) = \bar{s}_{\overline{1}|}\bar{a}_{\overline{1}|} = id/\delta^2 = 1.00028$, $\beta(\infty) = (\bar{s}_{\overline{1}|} - 1)/\delta = 0.50985$. Using other values available in Example 6.4.1 we find

$$10{,}000\ P^{(2)}(\bar{A}_{50:\overline{20}|}) = 329.69.$$

Then the increase in annual premium is

$$10{,}000[P^{[2]}(\bar{A}_{50:\overline{20}|}) - P^{(2)}(\bar{A}_{50:\overline{20}|})] = 1.01,$$

which is the annual benefit premium payable semiannually for the refund feature. ▼

6.6 Accumulation-Type Benefits

The analysis in this section is in terms of annual premiums for insurances payable at the end of the year of death. An analogous development is possible for fully continuous premiums and, with some adjustment, for semicontinuous premiums. We first seek the actuarial present value for an n-year term insurance on (x) for which the sum insured, in case death occurs in year $k + 1$, is $\ddot{s}_{\overline{k+1}|}$. The present-value random variable of this benefit, at policy issue, is

$$W = \begin{cases} v^{K+1}\ddot{s}_{\overline{K+1}|} = \dfrac{1}{d_{(j)}}[v^{K+1}(1+j)^{K+1} - v^{K+1}] & 0 \le K < n \\ 0 & K \ge n \end{cases}$$

where the insurer's present values are computed at interest rate i and $d_{(j)}$ is the discount rate equivalent to interest rate j. The actuarial present value is

$$\mathrm{E}[W] = \frac{A'^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|}}{d_{(j)}} \tag{6.6.1}$$

where $A'^{1}_{x:\overline{n}|}$ is calculated at the rate of interest $i' = (i - j)/(1 + j)$.

If $i = j$, then $i' = 0$, and the actuarial present value is

$$\begin{aligned} \frac{{}_nq_x - A^{1}_{x:\overline{n}|}}{d} &= \frac{1 - {}_np_x - A_{x:\overline{n}|} + v^n\,{}_np_x}{d} \\ &= \ddot{a}_{x:\overline{n}|} - {}_np_x\,\ddot{a}_{\overline{n}|} \\ &= \ddot{a}_{x:\overline{n}|} - {}_nE_x\,\ddot{s}_{\overline{n}|}. \end{aligned} \tag{6.6.2}$$

Formula (6.6.2) indicates that, when $j = i$, this special term insurance is equivalent to an n-year life annuity-due except for the event that (x) survives the n years. Then the term insurance would provide a benefit of zero, whereas the life annuity payments, given survival for n years, would have value $\ddot{s}_{\overline{n}|}$ at time n.

Now let us consider the situation where (x) has the choice of purchasing an n-year unit endowment insurance with an annual premium of $P_{x:\overline{n}|}$ or of establishing a savings fund with deposits of $1/\ddot{s}_{\overline{n}|}$ at the beginning of each of n years and purchasing a special decreasing term insurance. The special insurance will provide, in the event of death in year $k + 1$, the difference,

$$1 - \frac{\ddot{s}_{\overline{k+1}|}}{\ddot{s}_{\overline{n}|}} \qquad k = 0, 1, 2, \ldots, n - 1,$$

between the unit benefit under the endowment insurance and the accumulation in the savings fund. We suppose further that the same interest rate i is applicable in

valuing all these transactions. The same benefits are provided by the endowment insurance and by the combination of the special term insurance and the savings fund. Therefore one would anticipate that

(the annual benefit premium $P_{x:\overline{n}|}$ for the endowment insurance) = (the annual benefit premium for the special term insurance) + (the annual savings fund deposit $1/\ddot{s}_{\overline{n}|}$).

To verify this conjecture, we consider the present-value random variable for the special decreasing term insurance,

$$\tilde{W} = \begin{cases} v^{K+1}\left(1 - \dfrac{\ddot{s}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|}}\right) = v^{K+1} - \dfrac{\ddot{a}_{\overline{K+1}|}}{\ddot{s}_{\overline{n}|}} & 0 \le K < n \\ 0 & K \ge n. \end{cases} \tag{6.6.3}$$

The actuarial present value of $\tilde{W}$ is denoted by $\tilde{A}^1_{x:\overline{n}|}$ and given by

$$\begin{aligned} \tilde{A}^1_{x:\overline{n}|} &= \mathrm{E}[\tilde{W}] \\ &= A^1_{x:\overline{n}|} - \frac{\ddot{a}_{x:\overline{n}|} - {}_np_x\,\ddot{a}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}} \\ &= A^1_{x:\overline{n}|} - \frac{\ddot{a}_{x:\overline{n}|} - {}_nE_x\,\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{n}|}} \end{aligned}$$

[see (6.6.2)].

The annual benefit premium for the special term insurance is therefore

$$\begin{aligned} \tilde{P}^1_{x:\overline{n}|} = \frac{\tilde{A}^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} &= P^1_{x:\overline{n}|} - \frac{1}{\ddot{s}_{\overline{n}|}} + P_{x:\overline{n}|} \\ &= P_{x:\overline{n}|} - \frac{1}{\ddot{s}_{\overline{n}|}}, \end{aligned}$$

and then

$$P_{x:\overline{n}|} = \tilde{P}^1_{x:\overline{n}|} + \frac{1}{\ddot{s}_{\overline{n}|}}. \tag{6.6.4}$$

We have already seen that

$$P_{x:\overline{n}|} = P^1_{x:\overline{n}|} + P_{x:\overline{n}|},$$

and now (6.6.4) provides an alternative decomposition of $P_{x:\overline{n}|}$. The components are the annual premium for the special term insurance and the annual savings fund deposits, $1/\ddot{s}_{\overline{n}|}$, which accumulate to one at the end of n years.

Example 6.6.1

Derive formulas for the annual benefit premium for a 5,000, 20-year term insurance on (x) providing, in case of death within the 20 years, the return of the annual benefit premiums paid:

a. Without interest
b. Accumulated at the interest rate used in the determination of premiums.

In each case, the return of premiums is in addition to the 5,000 sum insured and benefit payments are made at the end of the year of death.

Solution:

a. Let π_a be the benefit premium. Then

$$\pi_a \, \ddot{a}_{x:\overline{20|}} = 5{,}000 \, A^1_{x:\overline{20|}} + \pi_a (IA)^1_{x:\overline{20|}}$$

and

$$\pi_a = 5{,}000 \, \frac{A^1_{x:\overline{20|}}}{\ddot{a}_{x:\overline{20|}} - (IA)^1_{x:\overline{20|}}}.$$

b. Let π_b be the benefit premium. We use (6.6.2) to obtain

$$\pi_b \, \ddot{a}_{x:\overline{20|}} = 5{,}000 \, A^1_{x:\overline{20|}} + \pi_b (\ddot{a}_{x:\overline{20|}} - {}_{20}E_x \, \ddot{s}_{\overline{20|}}),$$

$$\pi_b \, {}_{20}E_x \ddot{s}_{\overline{20|}} = 5{,}000 \, A^1_{x:\overline{20|}},$$

$$\pi_b = 5{,}000 \, \frac{A^1_{x:\overline{20|}}}{{}_{20}E_x \, \ddot{s}_{\overline{20|}}}$$

$$= 5{,}000 \, \frac{A^1_{x:\overline{20|}}}{{}_{20}p_x \, \ddot{a}_{\overline{20|}}}.$$

In practice, annual contract premiums would be refunded, and the formulas would take this into account. ▼

Example 6.6.2

A deferred annuity issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death during the premium paying period is the return of annual benefit premiums accumulated with interest at the rate used for the premium. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium.

Solution:

Equating the actuarial present value of the annual benefit premiums, π, to the actuarial present value of the benefits, we have

$$\pi \ddot{a}_{x:\overline{n|}} = {}_nE_x \, \ddot{a}_{x+n} + \pi(\ddot{a}_{x:\overline{n|}} - {}_nE_x \, \ddot{s}_{\overline{n|}})$$

where the second term on the right-hand side comes from (6.6.2). Solving for π yields

$$\pi = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n|}}}.$$ ▼

6.7 Notes and References

Lukacs (1948) provides a survey of the development of the equivalence principle. Premiums derived by an application of the equivalence principle are often called actuarial premiums in the literature of the economics of uncertainty. Gerber (1976, 1979) discussed exponential premiums and reserves; these were illustrated in Example 6.1.1 under Principle III. Fractional premiums of various types are important in practice. Scher (1974) has discussed developments in this area, namely, the relations among fully continuous, apportionable, and semicontinuous premiums. The decomposition of an endowment insurance premium appeared in a paper by Linton (1919).

Exercises

Section 6.1

6.1. Calculate the expectation and the variance of the present value of the financial loss for the insurance in Example 6.1.1, when the premium is determined by Principle I.

6.2. Verify that the exponential premium (with $\alpha = 0.1$) for the insurance in Example 6.1.1, modified so that the benefit amount is 10, is 3.45917. (Note that this is roughly 11.3 times as large as the exponential premium for a benefit amount of 1 found in Example 6.1.1.)

6.3. Using the assumptions of Example 6.1.1, determine the annual premium that maximizes the expected utility of an insurer with initial wealth $w = 10$ and utility function $u(x) = x - 0.01x^2$, $x < 50$. [Hint: Use (1.3.6), $w - 0.01w^2 = \mathrm{E}[(w - L) - 0.01(w - L)^2]$.]

Section 6.2

6.4. A fully continuous whole life insurance with unit benefit has a level premium. The time-until-death random variable, $T(x)$, has an exponential distribution with $\mathrm{E}[T(x)] = 50$ and the force of interest is $\delta = 0.06$.
a. If the principle of equivalence is used, find the benefit premium rate.
b. Find the premium rate if it is to be such that $\Pr(L > 0) = 0.50$.
c. Repeat part (b) if the force of interest, δ, equals 0.

6.5. If the force of mortality strictly increases with age, show that $\bar{P}(\bar{A}_x) > \mu_x(0)$. [Hint: Show that $\bar{P}(\bar{A}_x)$ is a weighted average of $\mu_x(t)$, $t > 0$.]

6.6. Following Example 6.2.1, derive a general expression for

$$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta \bar{a}_x)^2}$$

where $\mu_x(t) = \mu$ and δ is the force of interest for $t > 0$.

6.7. If $\delta = 0$, show that

$$\bar{P}(\bar{A}_x) = \frac{1}{\mathring{e}_x}.$$

6.8. Prove that the variance of the loss associated with a single premium whole life insurance is less than the variance of the loss associated with an annual premium whole life insurance. Assume immediate payment of claims on death and continuous payment of benefit premiums.

6.9. Show that

$$\left(1 + \frac{d\bar{a}_x}{dx}\right) \bar{P}(\bar{A}_x) - \frac{d\bar{A}_x}{dx} = \mu(x).$$

Section 6.3

6.10. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate values for the annual premiums in the following table. Note any patterns of inequalities that appear in the matrix of results.

Fully Continuous	Semicontinuous	Fully Discrete
$\bar{P}(\bar{A}_{35:\overline{10}\rceil})$	$P(\bar{A}_{35:\overline{10}\rceil})$	$P_{35:\overline{10}\rceil}$
$\bar{P}(\bar{A}_{35:\overline{30}\rceil})$	$P(\bar{A}_{35:\overline{30}\rceil})$	$P_{35:\overline{30}\rceil}$
$\bar{P}(\bar{A}_{35:\overline{60}\rceil})$	$P(\bar{A}_{35:\overline{60}\rceil})$	$P_{35:\overline{60}\rceil}$
$\bar{P}(\bar{A}_{35})$	$P(\bar{A}_{35})$	P_{35}
$\bar{P}(\bar{A}^1_{35:\overline{30}\rceil})$	$P(\bar{A}^1_{35:\overline{30}\rceil})$	$P^1_{35:\overline{30}\rceil}$
$\bar{P}(\bar{A}^1_{35:\overline{10}\rceil})$	$P(\bar{A}^1_{35:\overline{10}\rceil})$	$P^1_{35:\overline{10}\rceil}$

6.11. Show that

$${}_{20}P^1_{x:\overline{30}\rceil} - P^1_{x:\overline{20}\rceil} = {}_{20}P({}_{20|10}A_x).$$

6.12. Generalize Example 6.3.1 where

$${}_{k|}q_x = (1 - r)r^k \qquad k = 0, 1, 2, \ldots;$$

that is, derive expressions in terms of r and i for A_x, $\ddot{a}_x$, P_x, and $[{}^2A_x - (A_x)^2]/(d\,\ddot{a}_x)^2$.

Section 6.4

6.13. Using the information given in Example 6.4.1, calculate the value $P^{(2)}_{50}$.

6.14. Using various formulas for $\ddot{a}^{(m)}_{x:\overline{n}\rceil}$ under the assumption of a uniform distribution of deaths in each year of age, show that the ratio

$$\frac{\ddot{a}_{x:\overline{n}\rceil}}{\ddot{a}^{(m)}_{x:\overline{n}\rceil}}$$

in (6.4.2) can be expressed as the reciprocal of

a. $\ddot{a}^{(m)}_{\overline{1}|} - \beta(m)P^{1}_{x:\overline{n}|}$

b. $\alpha(m) - \beta(m)(P^{1}_{x:\overline{n}|} + d)$

c. $1 - \dfrac{m-1}{2m}(P^{1}_{x:\overline{n}|} + d)$.

6.15. Refer to Example 6.4.1(b) and directly calculate

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}}$$

using the Illustrative Life Table for the actuarial present values in the numerator and the denominator.

6.16. If

$$\frac{P^{1\,(12)}_{x:\overline{20}|}}{P^{1}_{x:\overline{20}|}} = 1.032$$

and $P_{x:\overline{20}|} = 0.040$, what is the value of $P^{(12)}_{x:\overline{20}|}$?

Section 6.5

6.17. Arrange in order of magnitude and indicate your reasoning:
$P^{(2)}(\bar{A}_{40:\overline{25}|})$, $\bar{P}(\bar{A}_{40:\overline{25}|})$, $P^{\{4\}}(\bar{A}_{40:\overline{25}|})$, $P(\bar{A}_{40:\overline{25}|})$, $P^{\{12\}}(\bar{A}_{40:\overline{25}|})$.

6.18. Given that

$$\frac{d}{d^{(12)}} = \frac{99}{100},$$

evaluate

$$\frac{P^{\{12\}}(\bar{A}_x)}{P^{\{1\}}(\bar{A}_x)}.$$

6.19. If $\bar{P}(\bar{A}_x) = 0.03$, and if interest is at the effective annual rate of 5%, calculate the semiannual benefit premium for a 50,000 whole life insurance on (x) where premiums are apportionable.

6.20. Show that

$$P^{\{m\}}(\bar{A}_{x:\overline{n}|}) - P^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}_{x:\overline{n}|} - A^{(m)}_{x:\overline{n}|}}{\delta\,\ddot{a}^{(m)}_{x:\overline{n}|}}\right)$$

$$= \bar{P}(\bar{A}_{x:\overline{n}|})\left(\frac{\bar{A}^{1}_{x:\overline{n}|} - A^{1(m)}_{x:\overline{n}|}}{\delta\,\ddot{a}^{(m)}_{x:\overline{n}|}}\right).$$

Section 6.6

6.21. Express

$$1 - \frac{\ddot{s}_{\overline{20}|}}{\ddot{s}_{45:\overline{20}|}}$$

as an annual premium. Interpret your result.

6.22. On the basis of the Illustrative Life Table and an interest rate of 6%, calculate the components of the two decompositions

a. $1{,}000\ P_{50:\overline{20}|} = 1{,}000(P^{1}_{50:\overline{20}|} + P_{50:\overline{20}|}^{\frac{1}{}})$

b. $1{,}000\ P_{50:\overline{20}|} = 1{,}000\left(\tilde{P}^{1}_{50:\overline{20}|} + \frac{1}{\ddot{s}_{\overline{20}|}}\right)$.

6.23. Consider the continuous random variable analogue of (6.6.3),

$$\tilde{\tilde{W}} = \begin{cases} v^{T}\left(1 - \dfrac{\bar{s}_{\overline{T}|}}{\bar{s}_{\overline{n}|}}\right) & 0 \le T < n \\ 0 & T \ge n. \end{cases}$$

The loss,

$$L = \tilde{\tilde{W}} - \tilde{\tilde{A}}^{1}_{x:\overline{n}|},$$

can be used with the equivalence principle to determine $\tilde{\tilde{A}}^{1}_{x:\overline{n}|}$, the single benefit premium for this special policy. Show that

a. $\tilde{\tilde{A}}^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} - \dfrac{\bar{a}_{x:\overline{n}|} - {}_{n}p_{x}\,\bar{a}_{\overline{n}|}}{\bar{s}_{\overline{n}|}}$

b. $\mathrm{E}[\tilde{\tilde{W}}^{2}] = \dfrac{(1+i)^{2n}\,{}^{2}\bar{A}^{1}_{x:\overline{n}|} - 2(1+i)^{n}\,\bar{A}^{1}_{x:\overline{n}|} + (1 - {}_{n}p_{x})}{[(1+i)^{n} - 1]^{2}}$.

Miscellaneous

6.24. Express

$$A_{40}\,P_{40:\overline{25}|} + (1 - A_{40})P_{40}$$

as an annual benefit premium. Interpret your result.

6.25. a. Show that

$$\frac{1}{\ddot{a}_{65:\overline{10}|}} - \frac{1}{\ddot{s}_{65:\overline{10}|}} = P^{1}_{65:\overline{10}|} + d.$$

b. What is the corresponding formula for

$$\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}} - \frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}?$$

c. Show that the amount of annual income provided by a single benefit premium of 100,000 where

- The income is payable at the beginning of each month while (65) survives during the next 10 years, and

- The single premium is returned at the end of 10 years if (65) reaches age 75,

is given by

$$100{,}000\left(\frac{1}{\ddot{a}^{(12)}_{65:\overline{10}|}}-\frac{1}{\ddot{s}^{(12)}_{65:\overline{10}|}}\right)=100{,}000(\beta)$$

where (β) denotes the answer to part (b) of this exercise.

6.26. An insurance issued to (35) with level premiums to age 65 provides
- 100,000 in case the insured survives to age 65, and
- The return of the annual contract premiums with interest at the valuation rate to the end of the year of death if the insured dies before age 65.

If the annual contract premium G is 1.1π where π is the annual benefit premium, write an expression for π.

6.27. If ${}_{15}P_{45}=0.038$, $P_{45:\overline{15}|}=0.056$, and $A_{60}=0.625$, calculate $P^{1}_{45:\overline{15}|}$.

6.28. A 20-payment life policy is designed to return, in the event of death, 10,000 plus all contract premiums without interest. The return-of-premium feature applies both during the premium paying period and after. Premiums are annual and death claims are paid at the end of the year of death. For a policy issued to (x), the annual contract premium is to be 110% of the benefit premium plus 25. Express in terms of actuarial present-value symbols the annual contract premium.

6.29. Express in terms of actuarial present-value symbols the initial annual benefit premium for a whole life insurance issued to (25), subject to the following provisions:
- The face amount is to be one for the first 10 years and two thereafter.
- Each premium during the first 10 years is 1/2 of each premium payable thereafter.
- Premiums are payable annually to age 65.
- Claims are paid at the end of the year of death.

6.30. Let L_1 be the insurer's loss on a unit of whole life insurance, issued to (x) on a fully continuous basis. Let L_2 be the loss to (x) on a continuous life annuity purchased for a single premium of one. Show that $L_1 \equiv L_2$ and give an explanation in words.

6.31. An ordinary life contract for a unit amount on a fully discrete basis is issued to a person age x with an annual premium of 0.048. Assume $d=0.06$, $A_x=0.4$, and ${}^{2}A_x=0.2$. Let L be the insurer's loss function at issue of this policy.

a. Calculate E[L].

b. Calculate Var(L).

c. Consider a portfolio of 100 policies of this type with face amounts given below.

Face Amount	Number of Policies
1	80
4	20

Assume the losses are independent and use a normal approximation to calculate the probability that the present value of gains for the portfolio will exceed 20.

6.32. Express, in terms of actuarial present-value symbols, the initial annual benefit premium for a unit of whole life insurance for (x) if after 5 years the annual benefit premium is double that payable during the first 5 years. Assume that death claims are made at the moment of death.

6.33. Repeat Exercise 6.20 for h-payment whole life insurance.

6.34. The function $l(t)$ is given by (6.2.1).
a. Establish that $l''(t) \geq 0$.
b. Adapt Jensen's inequality from Section 1.3 to establish that if $\bar{P} = \bar{P}(\bar{A}_x)$, then $\bar{P}(\bar{A}_x) \geq v^{\mathring{e}_x} / \bar{a}_{\overline{\mathring{e}_x}|}$.

6.35. If $T(x)$ has an exponential distribution with parameter μ,
a. Exhibit the p.d.f. of L as shown in (6.2.12)
b. Show that $E[L] = (\mu - \bar{P}) / (\mu + \delta)$.
c. Use part (b) to confirm that $E[L] = 0$ when $\bar{P} = \bar{P}(\bar{A}_x)$.

6.36. Use the assumptions of Exercise 6.35, with $\mu = 0.03$ and $\delta = 0.06$.
a. Evaluate $\Pr(L \leq 0)$ when $\bar{P} = \bar{P}(\bar{A}_x)$.
b. Determine $\bar{P}$ so that $\Pr(L > 0) = 0.5$.



BENEFIT RESERVES

7.1 Introduction

In Chapter 6 we introduced several principles that can be used for the determination of benefit premiums. The equivalence principle was used extensively in our discussion in Chapter 6. By it, an equivalence relation is established on the date a long-term contract is entered into by two parties who agree to exchange a set of payments. For example, under an amortized loan, a borrower may pay a series of equal monthly payments equivalent to the single payment by a lender at the date of the loan. An insured may pay a series of benefit premiums to an insurer equivalent, at the date of policy issue, to the sum to be paid on the death of the insured, or on survival of the insured to the maturity date. An individual may purchase a deferred life annuity by means of level premiums payable to an annuity organization equivalent, at the date of contract agreement, to monthly payments by the annuity organization to the individual when that person survives beyond a specified age. Equivalence in the loan example is in terms of present value, whereas in the insurance and annuity examples it is an equivalence between two actuarial present values.

After a period of time, however, there will no longer be an equivalence between the future financial obligations of the two parties. A borrower may have payments remaining to be made, whereas the lending organization has already performed its responsibilities. In other settings both parties may still have obligations. The insured may still be required to pay further benefit premiums, whereas the insurer has the duty to pay the face amount on maturity or the death of the insured. In our deferred annuity example, the individual may have completed the payments, whereas the annuity organization still has monthly remunerations to make.

In this chapter we study payments in time periods beyond the date of initiation. For this, a balancing item is required, and this item is a liability for one of the parties and an asset for the other. In the loan case, the balancing item is the outstanding principal, an asset for the lender and a liability for the borrower. In the other two cases, if the individual continues to survive, the balancing item is called

a reserve. This is typically a liability that should be recognized in any financial statement of an insurer or annuity organization, as the case may be. It is also typically an asset for the insured or individual purchasing the annuity.

We illustrate the determination of the balancing item spoken of above by continuation of Example 6.1.1 in the two cases where a utility function was used to define the premium principle.

Example 7.1.1

An insurer has issued a policy paying 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. Assume that the insured is still alive 1 year after entering into the contract. Further, assume that the insurer continues to use $i = 0.06$ and the following mortality assumption for K:

$$ {}_{k|}q_0 = 0.2 \qquad k = 0, 1, 2, 3, 4. $$

Find the reserve, ${}_1V$, as determined by the following:

a. Principle II: The insurer, using a utility of wealth function $u(x) = x$, will be indifferent between continuing with the risk while receiving premiums of 0.30272 (from Example 6.1.1) and paying the amount ${}_1V$ to a reinsurer to assume the risk.

b. Principle III: The insurer, using a utility of wealth function $u(x) = -e^{-0.1x}$, will be indifferent between continuing with the risk while receiving premiums of 0.30628 (from Example 6.1.1) and paying the amount ${}_1V$ to a reinsurer to assume the risk.

Solution:

The conditional probability function for K, the curtate-future-lifetime, given that $K \geq 1$, is

$$ \Pr(K = k | K \geq 1) = \frac{\Pr(K = k)}{\Pr(K \geq 1)} = \frac{0.2}{0.8} = 0.25 \qquad k = 1, 2, 3, 4. $$

The present value at duration 1 of the insurer's future financial loss is ${}_1L = v^{(K-1)+1} - P\,\ddot{a}_{\overline{(K-1)+1}|}$, where P is the premium determined in Example 6.1.1.

a. According to (1.3.1), we seek the amount ${}_1V$ such that $u(w - {}_1V) = \mathrm{E}[u(w - {}_1L)|K \geq 1]$. By Principle II $u(x) = x$, so we have

$$ w - {}_1V = \mathrm{E}[w - {}_1L | K \geq 1] = w - \mathrm{E}[{}_1L | K \geq 1]. $$

Thus, Principle II is equivalent to requiring that ${}_1V$ be chosen so that ${}_1V = \mathrm{E}[{}_1L | K \geq 1]$. For this example, this requirement is

$$ \begin{aligned} {}_1V &= \sum_{k=1}^{4} (v^{(k-1)+1} - 0.30272\,\ddot{a}_{\overline{(k-1)+1}|}) \times \Pr(K = k | K \geq 1) \\ &= \sum_{k=1}^{4} v^{(k-1)+1} \Pr(K = k | K \geq 1) - 0.30272 \sum_{k=1}^{4} \ddot{a}_{\overline{(k-1)+1}|} \times \Pr(K = k | K \geq 1), \end{aligned} \tag{7.1.1} $$

which gives ${}_1V = 0.15111$ as shown in the following calculation.

Outcome k	Conditional Probability	Present Value (1 Year after Issue) of Future Obligations of: Insurer	Insured	Insurer's Prospective Loss
1	0.25	$v = 0.94340$	$P\,\ddot{a}_{\overline{1}\rceil} = 0.30272$	0.64067
2	0.25	$v^2 = 0.89000$	$P\,\ddot{a}_{\overline{2}\rceil} = 0.58831$	0.30169
3	0.25	$v^3 = 0.83962$	$P\,\ddot{a}_{\overline{3}\rceil} = 0.85773$	−0.01811
4	0.25	$v^4 = 0.79209$	$P\,\ddot{a}_{\overline{4}\rceil} = 1.11191$	−0.31981
Expected Value		0.86628	0.71517	0.15111

The actuarial present value of the insurer's prospective loss is

$$0.86628 - 0.71517 = 0.15111.$$

b. Again by (1.3.1) and now using the utility function in Principle III, we have

$$-e^{-0.1(w-{}_1V)} = \mathrm{E}[-e^{-0.1(w-{}_1L)}|K \geq 1] = -e^{-0.1w}\,\mathrm{E}[e^{0.1\,{}_1L}|K \geq 1].$$

Thus, Principle III is equivalent to requiring that ${}_1V$ be chosen so that $e^{0.1\,{}_1V} = \mathrm{E}[e^{0.1\,{}_1L}|K \geq 1]$ or that ${}_1V = 10 \log \mathrm{E}[e^{0.1\,{}_1L}|K \geq 1]$.

The following table summarizes the calculation of ${}_1V$ using the premium (0.30628) from part (c) of Example 6.1.1.

Outcome k	Conditional Probability	Insurer's Prospective Loss, ${}_1L$	$e^{(0.1\,{}_1L)}$
1	0.25	0.63712	1.06579
2	0.25	0.29477	1.02992
3	0.25	−0.02819	0.99718
4	0.25	−0.33287	0.96726

Thus, $\mathrm{E}[e^{0.1\,{}_1L}|K \geq 1] = (1.06579 + 1.02992 + 0.99718 + 0.96726)(0.25) = 1.01504$ and ${}_1V = (\log 1.01504)/0.1 = 0.14925$. ▼

Henceforth, benefit reserves will be based on benefit premiums as determined by the equivalence principle in part (a) of Example 7.1.1. Thus, the ***benefit reserve at time t*** is the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums, the conditioning event being survivorship of the insured to time t. The type of reserve found in part (b) of Example 7.1.1 is called an ***exponential reserve.***

The sections in Chapter 7 parallel sections of Chapter 6 on benefit premiums. We assume, as we do in Example 7.1.1, that the mortality and interest rates adopted at policy issue for the determination of benefit premiums continue to be appropriate and are used in the determination of benefit reserves.

7.2 Fully Continuous Benefit Reserves

We now develop the benefit reserves related to the fully continuous benefit premiums developed in Section 6.2 by application of the equivalence principle.

Let us consider reserves for a whole life insurance of 1 issued to (x) on a fully continuous basis with an annual benefit premium rate of $\bar{P}(\bar{A}_{[x]})$. The corresponding reserve for an insured surviving t years later is defined under the equivalence principle as the conditional expected value of the prospective loss at time t, given that (x) has survived to t. More formally, for $T(x) > t$ the prospective loss is

$$ {}_tL = v^{T(x)-t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{\overline{T(x)-t}|}. \tag{7.2.1} $$

The reserve, as a conditional expectation, is calculated using the conditional distribution of the future lifetime at t for a life selected at x, given it has survived to t. In International Actuarial Notation symbols this is

$$ \begin{aligned} {}_t\bar{V}(\bar{A}_{[x]}) &= \mathrm{E}[{}_tL|T(x) > t] \\ &= \mathrm{E}[v^{T(x)-t}|T(x) > t] - \bar{P}(\bar{A}_{[x]})\, \mathrm{E}[\bar{a}_{\overline{T(x)-t}|}|T(x) > t] \\ &= \bar{A}_{[x]+t} - \bar{P}(\bar{A}_{[x]})\, \bar{a}_{[x]+t}. \end{aligned} \tag{7.2.2} $$

If the attained age was the only given information at issue of the insurance at age x, or for some other reason an aggregate mortality table is used for the distribution of the future lifetime, then the conditional distribution of $T(x) - t$ is the same as the distribution of $T(x + t)$, and (7.2.2) in symbols is

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\bar{a}_{x+t}. \tag{7.2.3} $$

Formulas (7.2.2) and (7.2.3) state that

(the benefit reserve) = (the actuarial present value for the whole life insurance from age $x + t$)

− (the actuarial present value of future benefit premiums payable at an annual rate of $\bar{P}(\bar{A}_x)$).

The formulations of $\bar{P}(\bar{A}_x)$ and ${}_t\bar{V}(\bar{A}_x)$ are related. When $t = 0$, (7.2.3) yields ${}_0\bar{V}(\bar{A}_x) = 0$. This is a consequence of applying the equivalence principle as of the time the contract was established.

Remark on Notation (Restated):

In this book, we will simplify the appearance of the formulas by suppressing the select notation unless its use is necessary or helpful in the particular situation. The symbol $\mu_x(t)$ will be used for the force of mortality in the development of benefit reserves to reinforce the idea that the conditional distributions used in reserve calculations are derived from the distribution of $T(x)$.

Benefit reserves are defined in Section 7.1 as the conditional expectation of loss variables. In evaluating these conditional expected values in this section, the distribution of $T(x) - t$, given $T(x) \geq t$, was used. The interest rate and the distribution of $T(x)$ used with the equivalence principle at time $t = 0$ to determine the benefit premium were used again. The survival of the life (x) to time t was the only new information incorporated into the expected value calculation. A comprehensive reserve principle would require that all new information relevant to the loss variables and their distributions be incorporated into the reserve calculation. The objective of this requirement would be to estimate liabilities appropriate for the time that the valuation is made. In Chapters 7 and 8 the process of learning from experience to modify the assumptions under which benefit reserves are estimated are not studied.

By steps analogous to those used to obtain (6.2.6), we can determine the variance of ${}_tL$ as follows:

$$ {}_tL = v^{T(x)-t}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}, \tag{7.2.4} $$

thus

$$ \begin{aligned} \mathrm{Var}[{}_tL|T(x) > t] &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 \mathrm{Var}[v^{T(x)-t}|T(x) > t] \\ &= \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2]. \end{aligned} \tag{7.2.5} $$

Note the relation to (6.2.6) and that the development holds for all premium levels. It is not dependent on the equivalence principle.

Example 7.2.1

Follow up Example 6.2.1 by calculating ${}_t\bar{V}(\bar{A}_x)$ and Var $[{}_tL|T(x) > t]$.

Solution:

Since $\bar{A}_x$, $\bar{a}_x$, and $\bar{P}(\bar{A}_x)$ are independent of age x, (7.2.3) becomes

$$ {}_t\bar{V}(\bar{A}_x) = \bar{A}_x - \bar{P}(\bar{A}_x)\,\bar{a}_x = 0 \qquad t \geq 0. $$

In this case, future premiums are always equivalent to future benefits, and no reserve is needed.

Also, in this case, (7.2.5) reduces to

$$ \mathrm{Var}[{}_tL|T(x) > t] = \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 [{}^2\bar{A}_x - (\bar{A}_x)^2] = \mathrm{Var}(L) = 0.25, $$

as in Example 6.2.1. Here the variance of ${}_tL$ depends on neither the age x nor the duration t. ▼

Example 7.2.2

On the basis of De Moivre's law with $l_x = 100 - x$ and the interest rate of 6%, calculate

a. $\bar{P}(\bar{A}_{35})$

b. ${}_t\bar{V}(\bar{A}_{35})$ and $\text{Var}[{}_tL|T(x) > t]$ for $t = 0, 10, 20, \ldots, 60$.

Solution:

a. From $l_x = 100 - x$, we obtain ${}_tp_{35} = 1 - t/65$ and ${}_tp_{35}\,\mu(35 + t) = 1/65$ for $0 \le t < 65$. It follows that

$$\bar{A}_{35} = \int_0^{65} v^t \frac{1}{65}\,dt = \frac{\bar{a}_{\overline{65}|0.06}}{65} = 0.258047.$$

Then

$$\bar{P}(\bar{A}_{35}) = \frac{\delta\,\bar{A}_{35}}{1 - \bar{A}_{35}} = 0.020266.$$

b. At age $35 + t$, we have $\bar{A}_{35+t} = \bar{a}_{\overline{65-t}|}/(65 - t)$ and

$${}_t\bar{V}(\bar{A}_{35}) = \bar{A}_{35+t} - 0.020266\,\frac{1 - \bar{A}_{35+t}}{\log(1.06)}.$$

Further,

$${}^2\bar{A}_{35+t} = \int_0^{65-t} v^{2u} \frac{1}{65 - t}\,du = \frac{{}^2\bar{a}_{\overline{65-t}|}}{65 - t}$$

and, from (7.2.5),

$$\text{Var}[{}_tL|T(x) > t] = \left[1 + \frac{0.020266}{\log(1.06)}\right]^2 [{}^2\bar{A}_{35+t} - (\bar{A}_{35+t})^2].$$

Applying these formulas, we obtain the following results.

t	${}_t\bar{V}(\bar{A}_{35})$	$\text{Var}[{}_tL\|T(35) > t]$
0	0.0000	0.1187
10	0.0557	0.1201
20	0.1289	0.1173
30	0.2271	0.1073
40	0.3619	0.0861
50	0.5508	0.0508
60	0.8214	0.0097

Note the convergence of $\text{Var}[{}_tL|T(35) > t]$ toward zero. There is a rendezvous with certainty. ▼

The table in Example 7.2.2 provides the mean and the variance of the conditional distributions of ${}_tL$ for selected values of t. To gain more insight into the nature of

reserves, let us explore these distributions of ${}_tL$ in more depth. We previously studied the case for $t = 0$ at (6.2.11) following Example 6.2.3. From (7.2.1),

$$
\begin{aligned}
{}_tL &= v^{T(x)-t} - \bar{P}(\bar{A}_x)\,\bar{a}_{\overline{T(x)-t}|} \\
&= v^{T(x)-t}\left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}.
\end{aligned} \tag{7.2.6}
$$

If $\delta > 0$, ${}_tL$ is a decreasing function of $T(x) - t$ and lies in the interval

$$
-\frac{\bar{P}(\bar{A}_x)}{\delta} < {}_tL \leq 1. \tag{7.2.7}
$$

Using Figure 6.2.1 as a guide we can repeat the steps of (6.2.11) to establish the following relationship between $F_{T(x)}(u)$ and the d.f. of the conditional distribution of ${}_tL$, given $T(x) > t$, which we denote by $F_{{}_tL}(y)$. For a y in the interval given by (7.2.7),

$$
\begin{aligned}
F_{{}_tL}(y) &= \Pr[{}_tL \leq y | T(x) > t] \\
&= \Pr\left[v^{T(x)-t}\left[\frac{\delta + \bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta} \leq y \,\middle|\, T(x) > t\right] \\
&= \Pr\left[v^{T(x)-t} \leq \frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)} \,\middle|\, T(x) > t\right] \\
&= \Pr\left[T(x) \geq t - \frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right] \,\middle|\, T(x) > t\right] \\
&= \frac{\Pr[T(x) \geq t - (1/\delta)\{\log[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\}]}{\Pr[T(x) > t]} \qquad (7.2.8) \\
&= \frac{1 - F_{T(x)}(t - (1/\delta)\log\{[\delta y + \bar{P}(\bar{A}_x)]/[\delta + \bar{P}(\bar{A}_x)]\})}{1 - F_{T(x)}(t)}. \qquad (7.2.9)
\end{aligned}
$$

Differentiation of (7.2.9) with respect to y derives the p.d.f. for the conditional distribution of ${}_tL$, given $T(x) > t$:

$$
f_{{}_tL}(y) = \left\{\frac{1}{[\delta y + \bar{P}(\bar{A}_x)][1 - F_{T(x)}(t)]}\right\} f_{T(x)}\left(t - \frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right). \tag{7.2.10}
$$

For an aggregate mortality law the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$, so (7.2.8), (7.2.9), and (7.2.10) reduce to

$$
\begin{aligned}
F_{{}_tL}(y) &= \Pr\left[T(x+t) \geq -\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right] \qquad (7.2.11) \\
&= 1 - F_{T(x+t)}\left(-\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right), \qquad (7.2.12) \\
f_{{}_tL}(y) &= \frac{1}{[\delta y + \bar{P}(\bar{A}_x)]} f_{T(x+t)}\left(-\frac{1}{\delta}\log\left[\frac{\delta y + \bar{P}(\bar{A}_x)}{\delta + \bar{P}(\bar{A}_x)}\right]\right). \qquad (7.2.13)
\end{aligned}
$$

To illustrate these concepts we will extend Example 7.2.2.

Example 7.2.3

For the insurance contract and assumptions in Example 7.2.2:

a. Exhibit the formulas for the d.f. and the p.d.f. of the conditional distribution for ${}_tL$, given $T(x) > t$.

b. Display graphs of these conditional p.d.f.'s for $t = 0$, 20, 40, and 60.

Solution:

a. Since Example 7.2.2 specifies an aggregate mortality law, we use formulas (7.2.12) and (7.2.13). In Example 7.2.2,

$$F_{T(35+t)}(u) = \frac{u}{65 - t} \qquad \text{for } 0 \le u \le 65 - t$$

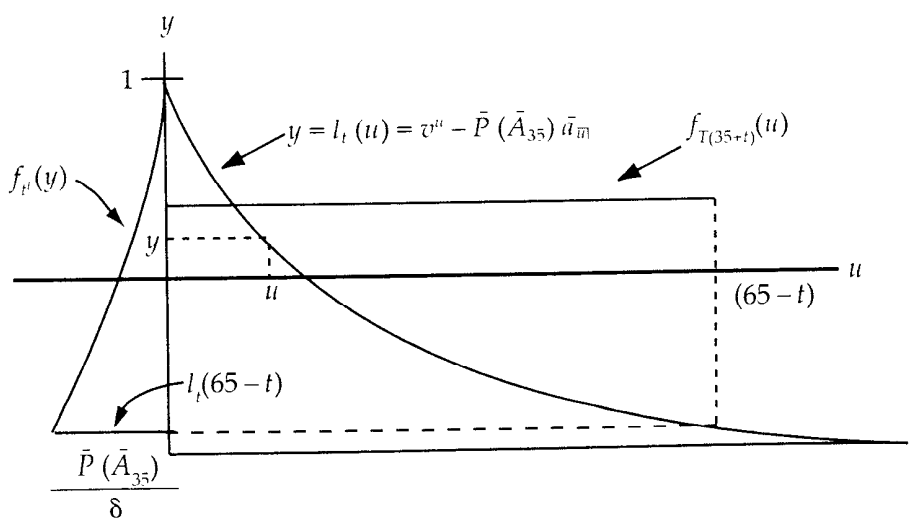
$$= 1 \qquad \text{for } 65 - t < u,$$

$$f_{T(35+t)}(u) = \frac{1}{65 - t} \qquad \text{for } 0 \le u \le 65 - t$$

$$= 0 \qquad \text{elsewhere.}$$

Figure 7.2.1 shows Figure 6.2.1 as it applies to this example. In this figure the outcome space of $T(35 + t)$ is on the u-axis, and the outcome space of the loss random variable, ${}_tL$, is on the y-axis. The relationship between the outcomes of $T(35 + t)$ and the outcomes of ${}_tL$ is given by the loss function $y = l_t(u)$ and is indicated by the dashed line connecting u and y in the figure. The p.d.f. $f_{T(35+t)}(u)$ has its domain on the u-axis and its range on the y-axis. The domain of the p.d.f. $f_{tL}(y)$ is on the y-axis, and its range is to be imagined on an axis perpendicular to the u-y plane, but for the sketch it has been laid perpendicular to the y-axis in the u-y plane.

Schematic Diagrams of $l_t(u)$, $f_{T(35+t)}(u)$, and $f_{tL}(y)$



To determine the d.f. by ${}_tL$ we start with a value of y corresponding to a value of u in the interval $(0, 65 - t)$. For such a y we have, by (7.2.12),

$$F_{tL}(y) = 1 - \frac{(-1/\delta)\log\{[\delta y + \bar{P}(\bar{A}_{35})]/[\delta + \bar{P}(\bar{A}_{35})]\}}{65 - t} \qquad 0 \le y \le 1.$$

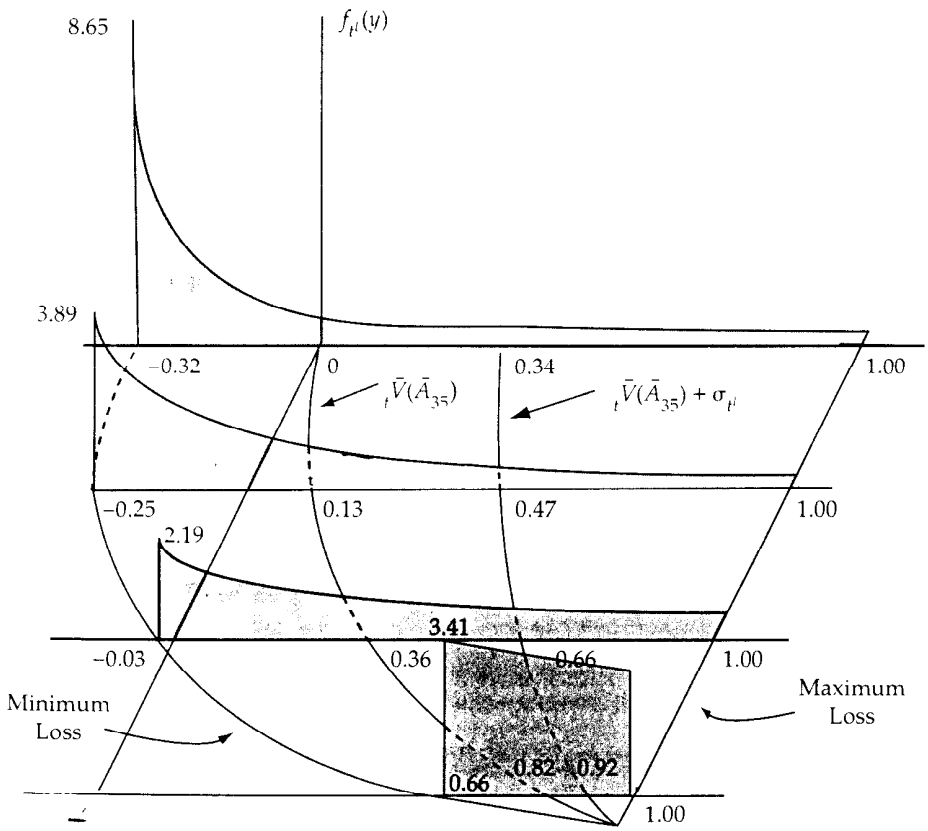
For a $y > 1$, $F_{tL}(y) = 1$.

Again, for a value of y corresponding to a value of u in the interval $(0, 65 - t)$, and using (7.2.13), we have

$$f_{tL}(y) = \begin{cases} \left(\frac{1}{65 - t}\right)\left[\frac{1}{\delta y + \bar{P}(\bar{A}_{35})}\right] & -\frac{\bar{P}(\bar{A}_{35})}{\delta} \le y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

b. Figure 7.2.2 is the composite of the required graphs of the p.d.f.'s $f_{tL}(y)$, for $t = 0$, 20, 40, and 60. Figure 7.2.1 is a graph for one value of t. Compare Figures 7.2.1 and 7.2.2 as follows: The vertical y-axis of Figure 7.2.1 is the horizontal axis in Figure 7.2.2. The axis that was imagined to be perpendicular to the u-y plane in Figure 7.2.1 is the vertical axis in Figure 7.2.2. Note the curves in the y-t plane that indicate the minimum and maximum possible losses, the expected loss (benefit reserve), and the expected loss plus one standard deviation of the loss. ▼

$f_{tL}(y)$ for $t = 0$, 20, 40, and 60



Corresponding to Table 6.2.1, Table 7.2.1 for benefit reserves is presented. We have not tabulated details of the prospective loss, ${}_tL$, and explicit formulas for $\mathrm{Var}[{}_tL|T(x) > t]$, corresponding to the several benefit reserves, are not displayed.



Fully Continuous Benefit Reserves; Age at Issue x, Duration t, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula
Whole life insurance	${}_t\bar{V}(\bar{A}_x)$	$\bar{A}_{x+t} - \bar{P}(\bar{A}_x)\,\bar{a}_{x+t}$
n-Year term insurance	${}_t\bar{V}(\bar{A}^1_{x:\overline{n}\rceil})$	$\begin{cases} \bar{A}^1_{x+t:\overline{n-t}\rceil} - \bar{P}(\bar{A}^1_{x:\overline{n}\rceil})\,\bar{a}_{x+t:\overline{n-t}\rceil} & t < n \\ 0 & t = n \end{cases}$
n-Year endowment insurance	${}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil})$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rceil} - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\,\bar{a}_{x+t:\overline{n-t}\rceil} & t < n \\ 1 & t = n \end{cases}$
h-Payment years, whole life insurance	${}^h_t\bar{V}(\bar{A}_x)$	$\begin{cases} \bar{A}_{x+t} - {}_h\bar{P}(\bar{A}_x)\,\bar{a}_{x+t:\overline{h-t}\rceil} & t \le h \\ \bar{A}_{x+t} & t > h \end{cases}$
h-Payment years, n-year endowment insurance	${}^h_t\bar{V}(\bar{A}_{x:\overline{n}\rceil})$	$\begin{cases} \bar{A}_{x+t:\overline{n-t}\rceil} - {}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil})\,\bar{a}_{x+t:\overline{h-t}\rceil} & t \le h < n \\ \bar{A}_{x+t:\overline{n-t}\rceil} & h < t < n \\ 1 & t = n \end{cases}$
n-Year pure endowment	${}_t\bar{V}(A^1_{x:\overline{n}\rceil})$	$\begin{cases} A^1_{x+t:\overline{n-t}\rceil} - \bar{P}(A^1_{x:\overline{n}\rceil})\,\bar{a}_{x+t:\overline{n-t}\rceil} & t < n \\ 1 & t = n \end{cases}$
Whole life annuity	${}_t\bar{V}({}_n\bar{a}_x)$	$\begin{cases} {}_{n-t}|\bar{a}_{x+t} - \bar{P}({}_n\bar{a}_x)\,\bar{a}_{x+t:\overline{n-t}\rceil} & t \le n \\ \bar{a}_{x-t} & t > n \end{cases}$

7.3 Other Formulas for Fully Continuous Benefit Reserves

So far we have defined the benefit reserve as the conditional expectation of the prospective loss random variable and developed only one method to write formulas for fully continuous benefit reserves, namely, the ***prospective method,*** stating that the benefit reserve is the difference between the actuarial present values of future benefits and of future benefit premiums. From the prospective method, we can easily develop three other general formulas for policies with level benefits and level benefit premium rates. We illustrate these for the case of n-year endowment insurances.

The ***premium-difference formula*** for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil})$ is obtained by factoring $\bar{a}_{x+t:\overline{n-t}\rceil}$ out of the prospective formula for ${}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil})$:

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}\rceil}) &= \left[\frac{\bar{A}_{x+t:\overline{n-t}\rceil}}{\bar{a}_{x+t:\overline{n-t}\rceil}} - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\right]\bar{a}_{x+t:\overline{n-t}\rceil} \\
&= \left[\bar{P}(\bar{A}_{x+t:\overline{n-t}\rceil}) - \bar{P}(\bar{A}_{x:\overline{n}\rceil})\right]\bar{a}_{x+t:\overline{n-t}\rceil}. \qquad (7.3.1)
\end{aligned}
$$

Formula (7.3.1) exhibits the benefit reserve as the actuarial present value of a premium difference payable over the remaining premium-payment term. The

premium difference is obtained by subtracting the original annual benefit premium from the benefit premium for an insurance issued at the attained age $x + t$ for the remaining benefits.

A second formula is obtained by factoring the actuarial present value of future benefits out of the prospective formula. Thus, for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$ we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{x:\overline{n}|}) &= \left[1 - \bar{P}(\bar{A}_{x:\overline{n}|})\,\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{A}_{x+t:\overline{n-t}|}}\right]\bar{A}_{x+t:\overline{n-t}|} \\
&= \left[1 - \frac{\bar{P}(\bar{A}_{x:\overline{n}|})}{\bar{P}(\bar{A}_{x+t:\overline{n-t}|})}\right]\bar{A}_{x+t:\overline{n-t}|}. \qquad (7.3.2)
\end{aligned}
$$

This exhibits the benefit reserve as the actuarial present value of a portion of the remaining future benefits, that portion which is not funded by the future benefit premiums still to be collected. Note that $\bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the benefit premium required if the future benefits were to be funded from only the future benefit premiums, but $\bar{P}(\bar{A}_{x:\overline{n}|})$ is the benefit premium actually payable. Thus, $\bar{P}(\bar{A}_{x:\overline{n}|}) / \bar{P}(\bar{A}_{x+t:\overline{n-t}|})$ is the portion of future benefits funded by future benefit premiums. This is called a ***paid-up insurance formula,*** named from the paid-up insurance nonforfeiture benefit to be discussed in Chapter 16. Formulas analogous to (7.3.1) and (7.3.2) exist for a wide variety of benefit reserves.

A third expression is the ***retrospective formula.*** We develop this from a more general relationship. We have, from Exercise 4.12 and from formulas (5.2.18) and (5.2.19), for $t < n - s$,

$$
\bar{A}_{x+s:\overline{n-s}|} = \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\,\bar{A}_{x+s+t:\overline{n-s-t}|}
$$

and

$$
\bar{a}_{x+s:\overline{n-s}|} = \bar{a}_{x+s:\overline{t}|} + {}_tE_{x+s}\,\bar{a}_{x+s+t:\overline{n-s-t}|}.
$$

Substituting these expressions into the prospective formula for ${}_s\bar{V}(\bar{A}_{x:\overline{n}|})$, we obtain

$$
\begin{aligned}
{}_s\bar{V}(\bar{A}_{x:\overline{n}|}) &= \bar{A}^{\,1}_{x+s:\overline{t}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{t}|} \\
&\quad + {}_tE_{x+s}\,[\bar{A}_{x+s+t:\overline{n-t-s}|} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s+t:\overline{n-s-t}|}] \\
&= \bar{A}^{\,1}_{x+s:\overline{t}|} + {}_tE_{x+s}\,{}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}) - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{t}|}. \qquad (7.3.3)
\end{aligned}
$$

Thus the benefit reserves at the beginning and end of a t-year interval are connected by the following argument:

(the benefit reserve at the beginning of the interval) = (the actuarial present value at the beginning of the interval of benefits payable during the interval)

+ (the actuarial present value at the beginning of the interval of a pure endowment for the amount of the benefit reserve at the end of the interval)

− (the actuarial present value of benefit premiums payable during the interval).

The rearranged symbolic form,

$$ {}_s\bar{V}(\bar{A}_{x:\overline{n}|}) + \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x+s:\overline{n-t}|} = \bar{A}_{x+s:\overline{t}|} + {}_tE_{x+s}\,{}_{s+t}\bar{V}(\bar{A}_{x:\overline{n}|}), \tag{7.3.4} $$

shows that the actuarial present values of the insurer's resources and obligations are equal.

The retrospective formula is obtained from (7.3.4) by setting $s = 0$, noting that ${}_0\bar{V}(\bar{A}_{x:\overline{n}|}) = 0$ by the equivalence principle, and solving for ${}_t\bar{V}(\bar{A}_{x:\overline{n}|})$. Thus,

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \frac{1}{{}_tE_x}\,[\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{x:\overline{t}|} - \bar{A}^{1}_{x:\overline{t}|}]. $$

Further, $\bar{s}_{x:\overline{t}|} = \bar{a}_{x:\overline{t}|}/{}_tE_x$ so the formula reduces to

$$ {}_t\bar{V}(\bar{A}_{x:\overline{n}|}) = \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{s}_{x:\overline{t}|} - {}_t\bar{k}_x. \tag{7.3.5} $$

Here

$$ {}_t\bar{k}_x = \frac{\bar{A}^{1}_{x:\overline{t}|}}{{}_tE_x} \tag{7.3.6} $$

is called the ***accumulated cost of insurance.*** One notes that

$$ {}_t\bar{k}_x = \int_0^t \frac{v^s\,{}_sp_x\,\mu_x(s)\,ds}{v^t\,{}_tp_x} = \int_0^t \frac{(1+i)^{t-s}\,l_{x+s}\,\mu_x(s)\,ds}{l_{x+t}}. \tag{7.3.7} $$

This can be interpreted as the assessment against each of the l_{x+t} survivors to provide for the accumulated value of the death claims in the survivorship group between ages x and $x + t$. Thus, the reserve can be viewed as the difference between the benefit premiums, accumulated with interest and shared among only the survivors at age $x + t$, and the accumulated cost of insurance.

We conclude this section with some special formulas that express the whole life insurance benefit reserves in terms of a single actuarial function. Analogous formulas hold for n-year endowment insurance benefit reserves when benefit premiums are payable continuously for the n years. Because we used (5.2.8) to express $\bar{P}(\bar{A}_x)$ in terms of δ and either $\bar{a}_x$ or $\bar{A}_x$, we can use those ideas here to express ${}_t\bar{V}(\bar{A}_x)$ in terms of one of the actuarial functions $\bar{a}_x$, $\bar{A}_x$, or $\bar{P}(\bar{A}_x)$ and δ.

For an annuity function formula, substitute (6.2.9) and (5.2.8) into the prospective formula (7.2.3) to obtain

$$ {}_t\bar{V}(\bar{A}_x) = 1 - \delta\bar{a}_{x+t} - \left(\frac{1}{\bar{a}_x} - \delta\right)\bar{a}_{x+t} = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}. \tag{7.3.8} $$

Further, substituting (6.2.9) into the premium-difference formula, we have

$$
\begin{aligned}
{}_t\bar{V}(\bar{A}_{\overline{1}}) &= [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\,\bar{a}_{x+t} \\
&= \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}.
\end{aligned} \tag{7.3.9}
$$

Finally, we can rewrite (7.3.8) using $\bar{A}_{x+t} = 1 - \delta\bar{a}_{x+t}$ to obtain

$${}_t\bar{V}(\bar{A}_{\overline{1}}) = 1 - \frac{1 - \bar{A}_{x+t}}{1 - \bar{A}_x} = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}. \tag{7.3.10}$$

These last three formulas depend on relationship (5.2.8) between the annuity for the premium paying period and the insurance for the benefit period. Thus they are available only for whole life and endowment insurances where the premium-paying period and the benefit period are the same. Moreover, the frequency of premium payment must be the same as the "frequency" of benefit payment. We will see that apportionable premiums satisfy these requirements in their own way.

Remark:

Although benefit reserves are non-negative in most applications, there is no mathematical theorem that guarantees this property. Indeed, the reader can combine Exercise 4.2 and formula (7.3.10) for a quick verification that negative benefit reserves are a real possibility.

7.4 Fully Discrete Benefit Reserves

The benefit reserves of this section are for the insurances of Section 6.3 which have annual premium payments and payment of the benefit at the end of the year of death. As in Section 7.2 the underlying mortality assumption can be on a select or aggregate basis. We will display the formulas for the aggregate case, which has simpler notation. Let us consider a whole life insurance with benefit issued to (x) with benefit premium $P_{\overline{1}}$. Following the development in Section 7.2, for an insured surviving k years later, we define the benefit reserve, denoted by ${}_kV_{\overline{1}}$, as the conditional expectation of the prospective loss, ${}_kL$, at duration k. More precisely,

$${}_kL = v^{(K(x)-k)+1} - P_{\overline{1}}\,\ddot{a}_{\overline{(K(x)-k)+1}} \tag{7.4.1}$$

$${}_kV_{\overline{1}} = \mathrm{E}[{}_kL | K(x) = k, k+1, \ldots]. \tag{7.4.2}$$

The prospective formula for the benefit reserve is

$${}_kV_{\overline{1}} = A_{x+k} - P_{\overline{1}}\,\ddot{a}_{x+k}. \tag{7.4.3}$$

As in Section 7.2 this formula is the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

Analogous to (7.2.4), we have

$$\begin{aligned}\text{Var}[{}_kL|K(x) &= k, k+1, \ldots] \\ &= \text{Var}\left[v^{[K(x)-k]+1}\left(1+\frac{P_x}{d}\right)\middle| K(x) = k, k+1, \ldots\right] \\ &= \left(1+\frac{P_x}{d}\right)^2 \text{Var}[v^{[K(x)-k]+1}|K(x) = k, k+1, \ldots] \\ &= \left(1+\frac{P_x}{d}\right)^2 [{}^2A_{x+k} - (A_{x+k})^2]. \end{aligned} \tag{7.4.4}$$

Example 7.4.1

Follow up Example 6.4.1 by calculating ${}_kV_x$ and $\text{Var}[{}_kL|K(x) = k, k+1, \ldots]$.

Solution:

Here A_x, $\ddot{a}_x$, and P_x are independent of age x so that $A_{x+k} = A_x$ and

$${}_kV_x = A_x - P_x\,\ddot{a}_x = 0 \qquad k = 0, 1, 2, \ldots.$$

Also, from (7.4.4), $\text{Var}[{}_kL|K(x) = k, k+1, \ldots] = \text{Var}(L) = 0.2347$. ▼

The benefit reserve formulas tabulated in Table 7.4.1 correspond to the benefit premiums in Table 6.3.1 and are analogous to the benefit reserves in Table 7.2.1.

Fully Discrete Benefit Reserves; Age at Issue x, Duration k, Unit Benefit

Plan	International Actuarial Notation	Prospective Formula
Whole life insurance	${}_kV_x$	$A_{x+k} - P_x\,\ddot{a}_{x+k}$
n-Year term insurance	${}_kV^1_{x:\overline{n}\rceil}$	$\begin{cases} A_{x+k:\overline{n-k}\rceil}^{1} - P^1_{x:\overline{n}\rceil}\,\ddot{a}_{x+k:\overline{n-k}\rceil} & k < n \\ 0 & k = n \end{cases}$
n-Year endowment insurance	${}_kV_{x:\overline{n}\rceil}$	$\begin{cases} A_{x+k:\overline{n-k}\rceil} - P_{x:\overline{n}\rceil}\,\ddot{a}_{x+k:\overline{n-k}\rceil} & k < n \\ 1 & k = n \end{cases}$
h-Payment years, whole life insurance	${}^h_kV_x$	$\begin{cases} A_{x+k} - {}_hP_x\,\ddot{a}_{x+k:\overline{h-k}\rceil} & k < h \\ A_{x+k} & k \geq h \end{cases}$
h-Payment years, n-year endowment insurance	${}^h_kV_{x:\overline{n}\rceil}$	$\begin{cases} A_{x+k:\overline{n-k}\rceil} - {}_hP_{x:\overline{n}\rceil}\,\ddot{a}_{x+k:\overline{h-k}\rceil} & k < h < n \\ A_{x+k:\overline{n-k}\rceil} & h \leq k < n \\ 1 & k = n \end{cases}$
n-Year pure endowment	${}_kV^{\frac{1}{}}_{x:\overline{n}\rceil}$	$\begin{cases} A_{x+k:\overline{n-k}\rceil}^{\frac{1}{}} - P_{x:\overline{n}\rceil}^{\frac{1}{}}\,\ddot{a}_{x+k:\overline{n-k}\rceil} & k < n \\ 1 & k = n \end{cases}$
Whole life annuity	${}_kV({}_n|\ddot{a}_x)$	$\begin{cases} {}_{n-k}|\ddot{a}_{x+k} - P({}_n|\ddot{a}_x)\,\ddot{a}_{x+k:\overline{n-k}\rceil} & k < n \\ \ddot{a}_{x+k} & k \geq n \end{cases}$

Example 7.4.2

Determine an expression in actuarial present values and benefit premiums for the $\text{Var}[{}_kL|K(x) = k, k+1, \ldots]$ for a fully discrete n-year endowment insurance with a unit benefit.

Solution:

$$
\begin{aligned}
{}_kL &= v^{K(x)-k+1}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = k, k+1, \ldots, n-1 \\
&= v^{n-k}\left(1 + \frac{P_{x:\overline{n}|}}{d}\right) - \frac{P_{x:\overline{n}|}}{d} \qquad K(x) = n, n+1, \ldots,
\end{aligned}
$$

$$
\text{Var}[{}_kL|K(x) = k, k+1, \ldots] = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 [{}^2A_{x+k:\overline{n-k}|} - (A_{x+k:\overline{n-k}|})^2].
$$

▼

In cases other than whole life or endowment insurances with premiums payable throughout the insurance term, the expressions for the variance of the loss may be difficult to summarize in convenient notation.

Formulas similar to those of Section 7.3 can be developed for fully discrete benefit reserves. We illustrate these by writing the formulas for ${}_kV_{x:\overline{n}|}$ with a minimum of discussion. Verbal interpretations and algebraic developments closely parallel those for fully continuous benefit reserves.

The premium difference formula is

$$
{}_kV_{x:\overline{n}|} = (P_{x+k:\overline{n-k}|} - P_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{n-k}|}. \tag{7.4.5}
$$

The paid-up insurance formula is

$$
{}_kV_{x:\overline{n}|} = \left(1 - \frac{P_{x:\overline{n}|}}{P_{x+k:\overline{n-k}|}}\right) A_{x+k:\overline{n-k}|}. \tag{7.4.6}
$$

For the retrospective formula, we first establish a result analogous to (7.3.3), namely, for $h < n - j$,

$$
{}_jV_{x:\overline{n}|} = A_{x+j:\overline{h}|} - P_{x:\overline{n}|}\,\ddot{a}_{x+j:\overline{h}|} + {}_hE_{x+j}\,{}_{j+h}V_{x:\overline{n}|}. \tag{7.4.7}
$$

Then, if $j = 0$, we have, since ${}_0V_{x:\overline{n}|} = 0$,

$$
\begin{aligned}
{}_hV_{x:\overline{n}|} &= \frac{1}{{}_hE_x}(P_{x:\overline{n}|}\,\ddot{a}_{x:\overline{h}|} - A^1_{x:\overline{h}|}) \\
&= P_{x:\overline{n}|}\,\ddot{s}_{x:\overline{h}|} - {}_hk_x.
\end{aligned} \tag{7.4.8}
$$

Here the accumulated cost of insurance is ${}_hk_x = A^1_{x:\overline{h}|} / {}_hE_x$, and a survivorship group interpretation is possible.

An interesting observation follows from the retrospective formula for the benefit reserve. Let us consider two different policies issued to (x), each for a unit of

insurance during the first h years. Here, h is less than or equal to the shorter of the two premium-payment periods. The retrospective formula for the benefit reserve on policy one is

$$ {}_hV_{(1)} = P_{(1)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x $$

and that for the benefit reserve on policy two is

$$ {}_hV_{(2)} = P_{(2)}\, \ddot{s}_{x:\overline{h}|} - {}_hk_x. $$

It follows that

$$ {}_hV_{(1)} - {}_hV_{(2)} = (P_{(1)} - P_{(2)})\, \ddot{s}_{x:\overline{h}|}, \tag{7.4.9} $$

which shows that the difference in the two benefit reserves equals the actuarial accumulated value of the difference in the benefit premiums $P_{(1)} - P_{(2)}$. Since $1 / \ddot{s}_{x:\overline{h}|} = P^{\frac{1}{}}_{x:\overline{h}|}$, formula (7.4.9) can be rearranged as

$$ P_{(1)} - P_{(2)} = P^{\frac{1}{}}_{x:\overline{h}|}\, ({}_hV_{(1)} - {}_hV_{(2)}). \tag{7.4.10} $$

The difference in the benefit premiums is expressed as the benefit premium for an h-year pure endowment of the difference in the benefit reserves at the end of h years. Formula (6.3.10) is a special case of (7.4.10) with ${}_nV_{x:\overline{n}|} = 1$ and ${}^{n}_{n}V_x = A_{x+n}$. Another illustration of (7.4.10) is

$$ P_x = P^{1}_{x:\overline{n}|} + P^{\frac{1}{}}_{x:\overline{n}|}\, {}_nV_x \tag{7.4.11} $$

since ${}_nV^{1}_{x:\overline{n}|} = 0$.

As in the fully continuous case, there are special formulas for whole life and endowment insurance benefit reserves in the fully discrete case. Parallel to (7.3.8)–(7.3.10), we have, by use of the relations $A_y = 1 - d\, \ddot{a}_y$ and $1 / \ddot{a}_y = P_y + d$,

$$ {}_kV_x = 1 - d\, \ddot{a}_{x+k} - \left(\frac{1}{\ddot{a}_x} - d\right) \ddot{a}_{x+k} $$

$$ = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}, \tag{7.4.12} $$

$$ {}_kV_x = 1 - \frac{P_x + d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}, \tag{7.4.13} $$

and

$$ {}_kV_x = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}. \tag{7.4.14} $$

Similar special formulas also hold for fully discrete n-year endowment insurance benefit reserves, but not for insurance benefit reserves in general.

Fully discrete insurances provide instructive examples for the deterministic, or expected cash flow, model of the operations of benefit reserves. This is displayed in Examples 7.4.3 and 7.4.4.

Example 7.4.3

Assume that a 5-year term life insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and, as a by-product, obtain the benefit reserves.

Solution:

We first calculate the annual benefit premium $\pi = 1{,}000\ P^{1}_{50:\overline{5}|} = 6.55692$. Then the expected accumulation of funds for the group through the collection of benefit premiums, the crediting of interest, and the payment of claims is as stated in the following:

(1) Year h	(2) Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\,\pi$	(3) Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	(4) Expected Interest $(0.06)\,(3)_h$	(5) Expected Death Claims $1{,}000\,d_{50+h-1}$	(6) Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	(7) Expected Number of Survivors at End of Year l_{50+h}	(8) $1{,}000\ {}_hV^{1}_{50:\overline{5}|}$ $(6)_h/(7)_h$
1	586 903	586 903	35 214	529 884	92 233	88 979.11	1.04
2	583 429	675 662	40 540	571 432	144 770	88 407.68	1.64
3	579 682	724 452	43 467	616 416	151 503	87 791.26	1.73
4	575 640	727 143	43 629	665 065	105 707	87 126.20	1.21
5	571 280	676 987	40 619	717 606	0	86 408.60	0.00

Note that the benefit reserves derived in the table match those calculated by formula. For example, at duration 2 we have

$$1{,}000\ A^{1}_{52:\overline{3}|} = 20.09 \text{ and } \ddot{a}_{52:\overline{3}|} = 2.81391.$$

Then

$$1{,}000\ {}_2V^{1}_{50:\overline{5}|} = 20.09 - (6.55692)(2.81391) = 1.64 \quad (\text{1.54})$$

▼

Example 7.4.4

Assume that a 5-year endowment insurance of 1,000 is issued on a fully discrete basis to each member of a group of l_{50} persons at age 50. Trace the cash flow expected for this group on the basis of the Illustrative Life Table with interest at 6%, and as a by-product obtain the benefit reserves.

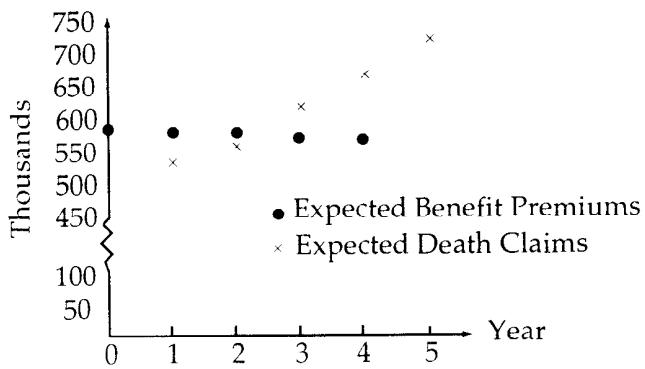
Solution:

Here the annual benefit premium is $\pi = 1{,}000\ P_{50:\overline{5}|} = 170.083$. The expected cash flow is displayed in the following table:

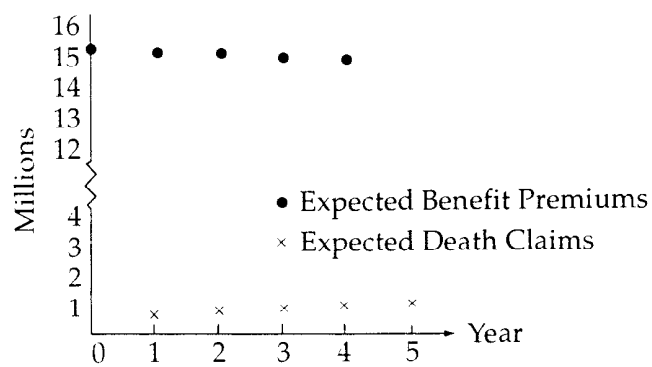
(1) Year h	(2) Expected Benefit Premiums at Beginning of Year $l_{50+h-1}\,\pi$	(3) Expected Fund at Beginning of Year $(2)_h + (6)_{h-1}$	(4) Expected Interest $(0.06)\,(3)_h$	(5) Expected Death Claims $1{,}000\,d_{50+h-1}$	(6) Expected Fund at End of Year $(3)_h + (4)_h - (5)_h$	(7) Expected Number of Survivors at End of Year l_{50+h}	(8) $1{,}000\,{}_hV_{50:\overline{5}|}^{1}$ $(6)_h \div (7)_h$
1	15 223 954	15 223 954	913 437	529 884	15 607 507	88 979.11	175.14
2	15 133 829	30 741 336	1 844 480	571 432	32 014 384	88 407.68	362.12
3	15 036 638	47 051 022	2 823 061	616 416	49 257 667	87 791.26	561.08
4	14 931 796	64 189 463	3 851 368	665 065	67 375 766	87 126.20	773.31
5	14 818 680	82 194 446	4 931 667	717 606	86 408 507	86 408.60	1 000.00

Figures 7.4.1 and 7.4.2 display the expected benefit premiums and expected death claims for the preceding two examples. In Example 7.4.3, expected benefit premiums exceed expected death claims for 2 years, but thereafter are less than expected claims. The excess benefit premiums accumulate a fund in the early years to be drawn on in the later years when expected claims are higher. At the end of 5 years, the fund is expected to be exhausted.

Expected Benefit Premiums and Expected Death Claims for Example 7.4.3



Expected Benefit Premiums and Expected Death Claims for Example 7.4.4



For the 5-year endowment case of Example 7.4.4, the picture is much different. As shown in Figure 7.4.2, the expected benefit premiums remain far in excess of expected death claims throughout. The expected fund at the end of 5 years is sufficient to provide 1,000 in maturity payments to each of the expected survivors.

The 5-year term insurance exemplifies a low-premium, low-accumulation life insurance, whereas the 5-year endowment insurance exemplifies a high-premium, high-accumulation form. Most life insurances would fall between these two extremes.

7.5 Benefit Reserves on a Semicontinuous Basis

We noted at the end of Section 6.3 that, in practice, there is a need for semicontinuous annual benefit premiums $P(\bar{A}_x)$, $P(\bar{A}_{x:\overline{n}|})$, $P(\bar{A}^1_{x:\overline{n}|})$, ${}_hP(\bar{A}_x)$, and ${}_hP(\bar{A}_{x:\overline{n}|})$ to take account of immediate payment of death claims. In such cases, the benefit reserve formulas in Table 7.4.1 need to be revised by replacement of A by $\bar{A}$ and of P by $P(\bar{A})$. Moreover, the principal symbol for the benefit reserve is now $V(\bar{A})$ with a subscript on the $\bar{A}$ to indicate the type of insurance as in the benefit premium symbol. For example, for an h-payment years, n-year endowment insurance

$$ {}_k^hV(\bar{A}_{x:\overline{n}|}) = \begin{cases} \bar{A}_{x+k:\overline{n-k}|} - {}_hP(\bar{A}_{x:\overline{n}|})\,\ddot{a}_{x+k:\overline{h-k}|} & k < h < n \\ \bar{A}_{x+k:\overline{n-k}|} & h \le k < n \\ 1 & k = n. \end{cases} \tag{7.5.1} $$

If a uniform distribution of deaths over each year of age is assumed, we have, from (4.4.2) and (6.3.12),

$$ {}_k^hV(\bar{A}_{x:\overline{n}|}) = \frac{i}{\delta}\,{}_k^hV^1_{x:\overline{n}|} + {}_k^hV_{x:\overline{n}|}. \tag{7.5.2} $$

Under this circumstance benefit reserves on a semicontinuous basis are easily calculated from the corresponding fully discrete benefit reserves.

7.6 Benefit Reserves Based on True *m*-thly Benefit Premiums

In this section we examine the benefit reserve formulas corresponding to the formulas for true m-thly benefit premiums discussed in Section 6.4. By the prospective method, one can write a direct formula for ${}_k^hV^{(m)}_{x:\overline{n}|}$, namely,

$$ {}_k^hV^{(m)}_{x:\overline{n}|} = A_{x+k:\overline{n-k}|} - {}_hP^{(m)}_{x:\overline{n}|}\,\ddot{a}^{(m)}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.6.1} $$

This can be evaluated after obtaining ${}_hP^{(m)}_{x:\overline{n}|}$ by means of (6.4.1) or (6.4.2), and $\ddot{a}^{(m)}_{x+k:\overline{h-k}|}$ by means of (5.4.15) or (5.4.17).

We now consider the difference between ${}_k^hV^{(m)}_{x:\overline{n}|}$ and ${}_k^hV_{x:\overline{n}|}$ in the general case of a limited payment endowment insurance. We have, for $k < h$,

$$
\begin{aligned}
{}_{k}^{h}V_{x:\overline{n}|}^{(m)} - {}_{k}^{h}V_{x:\overline{n}|} &= {}_{h}P_{x:\overline{n}|}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_{h}P_{x:\overline{n}|}^{(m)}\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)} \\
&= {}_{h}P_{x:\overline{n}|}^{(m)} \frac{\ddot{a}_{x:\overline{h}|}^{(m)}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} - {}_{h}P_{x:\overline{n}|}^{(m)}\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)}. \qquad (7.6.2)
\end{aligned}
$$

Under the assumption of a uniform distribution of deaths in each year of age, (7.6.2) becomes

$$
\begin{aligned}
{}_{k}^{h}V_{x:\overline{n}|}^{(m)} - {}_{k}^{h}V_{x:\overline{n}|} = {}_{h}P_{x:\overline{n}|}^{(m)} \Bigg\{ & \frac{\ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_{x:\overline{h}|} - \beta(m)\, A_{x:\overline{h}|}^{1}}{\ddot{a}_{x:\overline{h}|}}\, \ddot{a}_{x+k:\overline{h-k}|} \\
& - \left[\ddot{a}_{\overline{1}|}^{(m)}\, \ddot{a}_{x+k:\overline{n-k}|} - \beta(m)\, A_{x+k:\overline{h-k}|}^{1} \right] \Bigg\}.
\end{aligned}
$$

The terms involving $\ddot{a}_{\overline{1}|}^{(m)}$ cancel to yield

$$
{}_{k}^{h}V_{x:\overline{n}|}^{(m)} - {}_{k}^{h}V_{x:\overline{n}|} = \beta(m)\, {}_{h}P_{x:\overline{n}|}^{(m)}\, {}_{k}V_{x:\overline{h}|}^{1}. \qquad (7.6.3)
$$

Thus,

(the benefit reserve for an insurance with true m-thly benefit premiums) = (the corresponding fully discrete benefit reserve)

$$
+ \begin{pmatrix} \text{a fully discrete benefit reserve for term insurance over} \\ \text{the premium paying period for a fraction, } \beta(m), \text{ of the} \\ \text{true } m\text{-thly benefit premium for the plan of insurance} \end{pmatrix}.
$$

A similar result holds for benefit reserves on a semicontinuous basis with true m-thly benefit premiums under the assumption of uniform distribution of deaths in each year of age. By the prospective method, we have for $k < h$,

$$
{}_{k}^{h}V^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_{h}P^{(m)}(\bar{A}_{x:\overline{n}|})\, \ddot{a}_{x+k:\overline{h-k}|}^{(m)}. \qquad (7.6.4)
$$

By steps analogous to those connecting (7.6.1) and (7.6.3), we obtain

$$
{}_{k}^{h}V^{(m)}(\bar{A}_{x:\overline{n}|}) = {}_{k}^{h}V(\bar{A}_{x:\overline{n}|}) + \beta(m)\, {}_{h}P^{(m)}(\bar{A}_{x:\overline{n}|})\, {}_{k}V_{x:\overline{h}|}^{1}. \qquad (7.6.5)
$$

Further, by letting $m \to \infty$ above, we obtain for a fully continuous basis

$$
{}_{k}^{h}\bar{V}(\bar{A}_{x:\overline{n}|}) = {}_{k}^{h}V(\bar{A}_{x:\overline{n}|}) + \beta(\infty)\, {}_{h}\overline{P}(\bar{A}_{x:\overline{n}|})\, {}_{k}V_{x:\overline{h}|}^{1}. \qquad (7.6.6)
$$

Note again that the term insurance benefit reserve is on a fully discrete basis.

Example 7.6.1

On the basis of the Illustrative Life Table with the assumption of uniform distribution of deaths over each year of age and $i = 0.06$ calculate the following for a 20-year endowment insurance issued to (50) with a unit benefit and true semiannual benefit premiums:

a. The benefit reserve at the end of the tenth year if the benefit is payable at the end of the year of death.

b. The benefit reserve at the end of the tenth year if the benefit is payable at the moment of death.

Also verify (7.6.5) in relation to the benefit reserve in part (b).

Solution:

a. In addition to the values calculated in Example 6.4.1, we require

$$A^{1}_{60:\overline{10}|} = 0.13678852$$

$$A_{60:\overline{10}|} = 0.58798425$$

$$\ddot{a}_{60:\overline{10}|} = 7.2789425$$

$${}_{10}V^{1}_{50:\overline{20}|} = A^{1}_{60:\overline{10}|} - P^{1}_{50:\overline{20}|}\,\ddot{a}_{60:\overline{10}|} = 0.052752$$

$${}_{10}V_{50:\overline{20}|} = A_{60:\overline{10}|} - P_{50:\overline{20}|}\,\ddot{a}_{60:\overline{10}|} = 0.355380.$$

Then, under the assumption of a uniform distribution of deaths over each year of age, we have

$$\ddot{a}^{(2)}_{60:\overline{10}|} = \alpha(2)\,\ddot{a}_{60:\overline{10}|} - \beta(2)\,(1 - {}_{10}E_{60}) = 7.1392299.$$

The benefit reserve, ${}_{10}V^{(2)}_{50:\overline{20}|}$, can be calculated using either

(7.6.1): $$A_{60:\overline{10}|} - P^{(2)}_{50:\overline{20}|}\,\ddot{a}^{(2)}_{60:\overline{10}|} = 0.355822$$

or

(7.6.3): $${}_{10}V_{50:\overline{20}|} + \beta(2)\,P^{(2)}_{50:\overline{20}|}\,{}_{10}V^{1}_{50:\overline{20}|} = 0.355822.$$

b. We need additional calculated values:

$$\frac{i}{\delta}\,A^{1}_{50:\overline{20}|} = 0.13423835$$

$$P^{(2)}(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}^{(2)}_{50:\overline{20}|}} = 0.03286830$$

$$\frac{i}{\delta}\,A^{1}_{60:\overline{10}|} = 0.14085233$$

$$\bar{A}_{50:\overline{20}|} = 0.36471188$$

$$P(\bar{A}_{50:\overline{20}|}) = \frac{\bar{A}_{50:\overline{20}|}}{\ddot{a}_{50:\overline{20}|}} = 0.03229873$$

$$\bar{A}_{60:\overline{10}|} = 0.59204806$$

$${}_{10}V(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P(\bar{A}_{50:\overline{20}|})\,\ddot{a}_{60:\overline{10}|} = 0.3569475$$

$${}_{10}V^{(2)}(\bar{A}_{50:\overline{20}|}) = \bar{A}_{60:\overline{10}|} - P^{(2)}(\bar{A}_{50:\overline{20}|})\,\ddot{a}^{(2)}_{60:\overline{10}|} = 0.3573937$$

$$\beta(2)\,P^{(2)}(\bar{A}_{50:\overline{20}|})\,{}_{10}V^{1}_{50:\overline{20}|} = 0.000446.$$

This last value is the difference between the two directly above it, as shown in (7.6.5). ▼

7.7 Benefit Reserves on an Apportionable or Discounted Continuous Basis

In Section 6.5 we discussed apportionable, or discounted continuous, benefit premiums, and we now consider the corresponding benefit reserves. For integer k, we have by the prospective method

$$ {}_{k}^{h}V^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_{h}P^{(m)}(\bar{A}_{x:\overline{n}|})\, \ddot{a}^{(m)}_{x+k:\overline{h-k}|} \qquad k < h. \tag{7.7.1} $$

But by (6.5.2),

$$ {}_{h}P^{(m)}(\bar{A}_{x:\overline{n}|}) = \frac{d^{(m)}}{\delta}\, {}_{h}\bar{P}(\bar{A}_{x:\overline{n}|}), $$

and by (5.5.4),

$$ \ddot{a}^{(m)}_{x+k:\overline{h-k}|} = \frac{\delta}{d^{(m)}}\, \bar{a}_{x+k:\overline{h-k}|}. $$

Substitution into (7.7.1) yields, for an integer k,

$$ {}_{k}^{h}V^{(m)}(\bar{A}_{x:\overline{n}|}) = \bar{A}_{x+k:\overline{n-k}|} - {}_{h}\bar{P}(\bar{A}_{x:\overline{n}|})\, \bar{a}_{x+k:\overline{h-k}|} = {}_{k}^{h}\bar{V}(\bar{A}_{x:\overline{n}|}). \tag{7.7.2} $$

This means that, on anniversaries of the issue date, fully continuous benefit reserves can be used for all apportionable cases, independent of the premium-paying mode. The condition that k be an integer can be relaxed to being at the end of an m-th for m-thly premiums.

In Section 6.5, it was noted that the apportionable benefit premium could be decomposed as

$$ P^{(1)}(\bar{A}_x) = P(\bar{A}_x) + P(\bar{A}_x^{\mathrm{PR}}) \tag{7.7.3} $$

where the superscript PR is used to denote an insurance for the benefit premium refund feature. A similar decomposition for the benefit reserves can be verified by use of the prospective method and (6.5.7). The steps are:

$$ {}_{k}V(\bar{A}_x^{\mathrm{PR}}) = \bar{P}(\bar{A}_x)\, \frac{\bar{A}_{x+k} - A_{x+k}}{\delta} - P(\bar{A}_x^{\mathrm{PR}})\, \ddot{a}_{x+k} $$

$$ = \bar{P}(\bar{A}_x)\, \frac{d\ddot{a}_{x+k} - \delta\bar{a}_{x+k}}{\delta} - [P^{(1)}(\bar{A}_x) - P(\bar{A}_x)]\, \ddot{a}_{x+k}. $$

Since

$$ \frac{d}{\delta}\, \bar{P}(\bar{A}_x) = P^{(1)}(\bar{A}_x), $$

the expression can be reduced to

$$ {}_{k}V(\bar{A}_x^{\mathrm{PR}}) = -\bar{P}(\bar{A}_x)\, \bar{a}_{x+k} + P(\bar{A}_x)\, \ddot{a}_{x+k} $$

$$ = \bar{A}_{x+k} - \bar{P}(\bar{A}_x)\, \bar{a}_{x+k} - [\bar{A}_{x+k} - P(\bar{A}_x)\, \ddot{a}_{x+k}] $$

$$= {}_k\bar{V}(\bar{A}_x) - {}_kV(\bar{A}_x)$$

$$= {}_kV^{(1)}(\bar{A}_x) - {}_kV(\bar{A}_x).$$

Thus we have

$${}_kV^{(1)}(\bar{A}_x) = {}_kV(\bar{A}_k) + {}_kV(\bar{A}_x^{PR}). \tag{7.7.4}$$

7.8 Notes and References

This chapter has developed the idea of a reserve in parallel to the development of premiums in Chapter 6. Discussion of recursion formulas for reserves is deferred to Chapter 8. Reserve principles based on the utility functions used in Chapter 6 were first applied. Gerber (1976, 1979) develops these reserves in a more abstract setting. Benefit reserves, which followed from a linear utility function, were studied extensively. Scher (1974) explored the apportionable benefit premium reserves as discounted fully continuous benefit reserves.

Exercises

Section 7.1

7.1. Determine the benefit reserve for $t = 2, 3, 4$, and 5 for the insurance in Example 6.1.1.

7.2. Determine the exponential reserve for $t = 2, 3, 4$, and 5 for the insurance in Example 6.1.1.

7.3. Determine the exponential reserve for $t = 1, 2, 3, 4$, and 5 for the insurance in Exercise 6.2.

7.4. Consider the insurance in Example 7.1.1 and the insurer of Exercise 6.3 with utility function $u(x) = x - 0.01\,x^2$, $0 < x < 50$. Determine the reserve, ${}_kV$, for $k = 1, 2, 3$, and 4 such that the insurer, with wealth 10 at each duration, will be indifferent between continuing the risk while receiving premiums of 0.30360 (from Exercise 6.3) and paying the amount ${}_kV$ to a reinsurer to assume the risk.

7.5. Consider a unit insurance issued to (0) on a fully continuous basis using the following assumptions:
   i. De Moivre's law with $\omega = 5$
   ii. $i = 0.06$
   iii. Principle III of Example 6.1.1 with $\alpha = 0.1$.
   a. Display equations which can be solved for the exponential premium and the exponential reserve at $t = 1$.

b. Solve the equations of (a) for the numerical values for the exponential premium and exponential reserve. Numerical methods must be used to obtain these required solutions.

Section 7.2

7.6. For an n-year unit endowment insurance issued on a fully continuous basis to (x), define ${}_tL$, the prospective loss after duration t. Confirm that

$$\text{Var}({}_tL|T > t) = \frac{{}^2\bar{A}_{x+t:\overline{n-t}|} - (\bar{A}^2_{x+t:\overline{n-t}|})^2}{(\delta\bar{a}_{x:\overline{n}|})^2}.$$

7.7. The prospective loss, after duration t, for a single benefit premium n-year continuous temporary life annuity of 1 per annum issued to (x) is given by

$${}_tL = \begin{cases} \bar{a}_{\overline{T-t}|} & t \le T < n \\ \bar{a}_{\overline{n-t}|} & T \ge n. \end{cases}$$

Express $\text{E}[{}_tL|T > t]$ and $\text{Var}({}_tL|T > t)$ in symbols of actuarial present values.

7.8. Write prospective formulas for
a. ${}^{20}_{10}\bar{V}(\bar{A}_{35:\overline{30}|})$
b. the benefit reserve at the end of 5 years for a unit benefit 10-year term insurance issued to (45) on a single premium basis.

7.9. a. For the fully continuous whole life insurance with the benefit premium determined by the equivalence principle, determine the outcome $u_0 = T(x) - t$ such that the loss is zero. [Caution: For large values of t, a solution may not exist.]
b. Determine the value of u_0 for $t = 20$ in Example 7.2.3 and compare it to Figure 7.2.1 for reasonableness.

7.10. The assumptions of Example 7.2.3 are repeated. Find the value of t such that the minimum loss is zero. Check your result by examining Figure 7.2.2.

7.11. a. Repeat the development leading to (7.2.9) to obtain the d.f. for the loss variable associated with an n-year fully continuous endowment insurance.
b. Draw the sketch that corresponds to Figure 7.2.1 for this endowment insurance.

7.12. Repeat Exercise 7.11 for an n-year fully continuous term insurance.

7.13. Confirm that (7.2.10) satisfies the conditions for a p.d.f.

Section 7.3

7.14. Write four formulas for ${}^{20}_{10}\bar{V}(\bar{A}_{40})$.

7.15. Write seven formulas for ${}_{10}\bar{V}(\bar{A}_{40:\overline{20}|})$.

7.16. Give the retrospective formula for ${}^{30}_{20}\bar{V}({}_{30|}\bar{a}_{35})$.

7.17. For $0 < t \le m$, show

a. $\bar{P}(\bar{A}_{x:\overline{m+n}|}) = \bar{P}(\bar{A}^{1}_{x:\overline{m}|}) + \bar{P}_{x:\overline{m}|}^{\,1}\, {}_{m}\bar{V}(\bar{A}_{x:\overline{m+n}|})$

b. ${}_{t}\bar{V}(\bar{A}_{x:\overline{m+n}|}) = {}_{t}\bar{V}(\bar{A}^{1}_{x:\overline{m}|}) + {}_{t}\bar{V}_{x:\overline{m}|}^{\,1}\, {}_{m}\bar{V}(\bar{A}_{x:\overline{m+n}|})$

and give an interpretation in words.

7.18. State what formula in Section 7.3 the following equation is related to, and give an interpretation in words:

$$ {}^{20}_{10}\bar{V}(\bar{A}_{30}) = \bar{A}^{\,1}_{40:\overline{5}|} + {}_{5}E_{40}\, {}^{20}_{15}\bar{V}(\bar{A}_{30}) - {}_{20}\bar{P}(\bar{A}_{30})\, \bar{a}_{40:\overline{5}|}. $$

Section 7.4

7.19. Write four formulas for ${}^{20}_{10}V_{40}$.

7.20. Write seven formulas for ${}_{10}V_{40:\overline{20}|}$.

7.21. For $0 < k \le m$, show

$$ {}_{k}V_{x:\overline{m+n}|} = {}_{k}V^{1}_{x:\overline{m}|} + {}_{k}V_{x:\overline{m}|}^{\,1}\, {}_{m}V_{x:\overline{m+n}|}. $$

7.22. If $k < n/2$, ${}_{k}V_{x:\overline{n}|} = 1/6$, and $\ddot{a}_{x:\overline{n}|} + \ddot{a}_{x+2k:\overline{n-2k}|} = 2\,\ddot{a}_{x+k:\overline{n-k}|}$, calculate ${}_{k}V_{x+k:\overline{n-k}|}$.

Section 7.5

7.23. On the basis of the Illustrative Life Table and interest of 6%, calculate values for the benefit reserves in the following table. (See Exercise 6.10.)

Fully Continuous	Semicontinuous	Fully Discrete
${}_{10}\bar{V}(\bar{A}_{35:\overline{30}|})$	${}_{10}V(\bar{A}_{35:\overline{30}|})$	${}_{10}V_{35:\overline{30}|}$
${}_{10}\bar{V}(\bar{A}_{35})$	${}_{10}V(\bar{A}_{35})$	${}_{10}V_{35}$
${}_{10}\bar{V}(\bar{A}^{1}_{35:\overline{30}|})$	${}_{10}V(\bar{A}^{1}_{35:\overline{30}|})$	${}_{10}V^{1}_{35:\overline{30}|}$

7.24. Under the assumption of a uniform distribution of deaths in each year of age, which of the following are correct?

a. ${}_{k}V(\bar{A}_{x:\overline{n}|}) = \dfrac{i}{\delta}\, {}_{k}V_{x:\overline{n}|}$

b. ${}_{k}V(\bar{A}_{x}) = \dfrac{i}{\delta}\, {}_{k}V_{x}$

c. ${}_{k}V(\bar{A}^{1}_{x:\overline{n}|}) = \dfrac{i}{\delta}\, {}_{k}V^{1}_{x:\overline{n}|}$

Section 7.6

7.25. Show that, under the assumption of a uniform distribution of deaths in each year of age,

$$ \frac{{}_{5}V^{(4)}_{30:\overline{20}|} - {}_{5}V_{30:\overline{20}|}}{{}^{20}_{5}V^{(4)}_{30} - {}^{20}_{5}V_{30}} = \frac{A_{30:\overline{20}|}}{A_{30}}. $$

(The assumption is sufficient but not necessary.)

7.26. Which of the following are correct formulas for ${}_{15}V^{(m)}_{40}$?

a. $(P^{(m)}_{55} - P^{(m)}_{40})\,\ddot{a}^{(m)}_{55}$

b. $\left(1 - \dfrac{P^{(m)}_{40}}{P^{(m)}_{55}}\right) A_{55}$

c. $P^{(m)}_{40}\,\ddot{s}^{(m)}_{40:\overline{15}|} - {}_{15}k_{40}$

d. $1 - \dfrac{\ddot{a}^{(m)}_{55}}{\ddot{a}^{(m)}_{40}}$

Section 7.7

7.27. Which of the following are correct formulas for ${}_{15}V^{(4)}(\bar{A}_{40})$?

a. ${}_{15}\bar{V}(\bar{A}_{40})$

b. $[P^{(4)}(\bar{A}_{55}) - P^{(4)}(\bar{A}_{40})]\ddot{a}^{(4)}_{55}$

c. $[\bar{P}(\bar{A}_{55}) - \bar{P}(\bar{A}_{40})]\bar{a}_{55}$

d. $\left[1 - \dfrac{\bar{P}(\bar{A}_{40})}{\bar{P}(\bar{A}_{55})}\right] \bar{A}_{55}$

e. $1 - \dfrac{\bar{a}_{55}}{\bar{a}_{40}}$

f. $\bar{P}(\bar{A}_{40})\,\bar{s}_{40:\overline{15}|} - {}_{15}\bar{k}_{40}$

7.28. Show that

a. $P^{(m)}(\bar{A}_{x:\overline{n}|}) = {}_{n}P^{(m)}(\bar{A}_{x}) + (1 - \bar{A}_{x+n})P^{(m)}_{\frac{1}{x:\overline{n}|}}$

b. ${}_{k}V^{(m)}(\bar{A}_{x:\overline{n}|}) = {}^{n}_{k}V^{(m)}(\bar{A}_{x}) + (1 - \bar{A}_{x+n})\,{}_{k}V^{(m)}_{\frac{1}{x:\overline{n}|}}$.

Give an interpretation in words.

Miscellaneous

7.29. Calculate the value of $P^{1}_{x:\overline{n}|}$ if ${}_{n}V_{x} = 0.080$, $P_{x} = 0.024$, and $P_{\frac{1}{x:\overline{n}|}} = 0.2$.

7.30. If ${}_{10}V_{35} = 0.150$ and ${}_{20}V_{35} = 0.354$, calculate ${}_{10}V_{45}$.

7.31. A whole life insurance issued to (25) pays a unit benefit at the end of the year of death. Premiums are payable annually to age 65. The benefit premium for the first 10 years is P_{25} followed by an increased level annual benefit premium for the next 30 years. Use your Illustrative Life Table and $i = 0.06$ to find the following.

a. The annual benefit premium payable at ages 35 through 64.

b. The tenth-year benefit reserve.

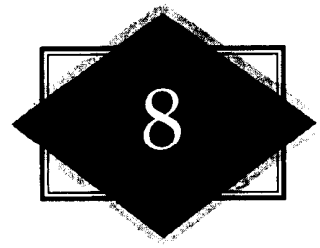
c. At the end of 10 years the policyholder has the option to continue with the benefit premium P_{25} until age 65 in return for reducing the death benefit to B for death after age 35. Calculate B.

d. If the option in (c) is selected, calculate the twentieth-year benefit reserve.

7.32. Assuming $\delta = 0.05$, $q_{x} = 0.05$, and a uniform distribution of deaths in each year of age, calculate

a. $(\bar{I}\bar{A})^{1}_{x:\overline{1}|}$

b. ${}_{1/2}V(\bar{I}\bar{A})^{1}_{x:\overline{1}|}$.



ANALYSIS OF BENEFIT RESERVES

8.1 Introduction

In Chapter 3 probability distributions for future lifetime random variables were developed. Chapters 4 and 5 studied the present-value random variables for insurances and annuities. The funding of insurance and annuities with a system of periodic payments was explored in Chapter 6, and in Chapter 7 the evolution of the liabilities under the periodic payments to fund an insurance or annuity was discussed. In these last two chapters the emphasis was on *level* benefits funded by *level* periodic payments that are usually determined by an application of the equivalence principle.

Why was the emphasis on level payments? First, traditional insurance products are purchased with level contract premiums. It is natural to think of a constant portion of each premium being for the benefit, hence a level benefit premium. Second, the single equation of the equivalence principle yields a solution for only one parameter. It is natural to think of this parameter as the benefit premium. Third, until the incidence of expenses is discussed in Chapter 15, one of the motivations of nonlevel benefit premiums is not present. Fourth, historically some regulatory standards have been specified in terms of benefit reserves defined by level benefit premiums.

In this chapter we define benefit reserves as we did in Chapter 7, but the definition is applied to general contracts with possibly nonlevel benefits and premiums. Of course, level premiums are a special case of nonlevel premiums, so the ideas here apply to the examples of Chapter 7. However, the reverse is not true. The special technique and relationships of Chapter 7 may not apply to the more general contracts of this chapter.

We start with definitions of general fully continuous and fully discrete insurances, and recursion relations are developed for these general models. The general discrete model is used to obtain formulas for benefit reserves at durations other than a contract anniversary, something that was not obtained in Chapter 7. An

allocation of loss and of risk of the contract to the various periods of the contract duration is obtained by use of the general fully discrete model. Again, these ideas apply to the contracts of Chapter 7, and the reader is encouraged to exercise this application.

8.2 Benefit Reserves for General Insurances

Consider a general fully discrete insurance on (x) in which

- The death benefit is payable at the end of the policy year of death
- Premiums are payable annually, at the beginning of the policy year
- The death benefit in the j-th policy year is b_j, $j = 1, 2, \ldots$
- The benefit premium payment in the j-th policy year is π_{j-1}, $j = 1, 2, \ldots$.

Note that the subscripts of b and π are the times of payment.

For a non-negative integer, h, the prospective loss, ${}_hL$, is the present value at h of the future benefits less the present value at h of the future benefit premiums. Expressed as a function of $K(x)$, it is

$$ {}_hL = \begin{cases} 0 & K(x) = 0, 1, \ldots h-1 \\ b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} & K(x) = h, h+1, \ldots . \end{cases} \tag{8.2.1} $$

Note: This definition extends the one given in (7.4.1) by including values (zeros) of ${}_hL$ for $K(x)$ less than h. Of course, this extension will not change the value of the benefit reserve because it is the conditional expectation given $K(x) \geq h$. The extension will be used in the development of recursion relations.

The benefit reserve at h, which we will denote by ${}_hV$, is defined as

$$ \begin{aligned} {}_hV &= \mathrm{E}[{}_hL | K(x) \geq h] \\ &= \mathrm{E}\left[b_{K(x)+1}\, v^{K(x)+1-h} - \sum_{j=h}^{K(x)} \pi_j\, v^{j-h} | K(x) \geq h \right] \\ &= \mathrm{E}\left[b_{(K(x)-h)+h+1}\, v^{(K(x)-h)+1} - \sum_{j=0}^{K(x)-h} \pi_{h+j}\, v^j | K(x) \geq h \right]. \end{aligned} \tag{8.2.2} $$

Under the assumption that the conditional distribution of $K(x) - h$, given $K(x) = h, h+1, \ldots$, is equal to the distribution of $K(x+h)$, this last expression can be rewritten as

$$ \begin{aligned} {}_hV &= \mathrm{E}\left[b_{K(x+h)+h+1}\, v^{K(x+h)+1} - \sum_{j=0}^{K(x+h)} \pi_{h+j}\, v^j \right] \\ &= \sum_{j=0}^{\infty} \left(b_{h+j+1}\, v^{j+1} - \sum_{k=0}^{j} \pi_{h+k}\, v^k \right) {}_jp_{x+h}\, q_{x+h+j}. \end{aligned} \tag{8.2.3} $$

Note that if this assumption fails, we are in the select mortality mode. By applying summation by parts (see Appendix 5) or reversing the order of summation, (8.2.3) can be rewritten as

$$ {}_hV = \sum_{j=0}^{\infty} b_{h+j+1}\, v^{j+1}\, {}_jp_{x+h}\, q_{x+h+j} - \sum_{j=0}^{\infty} \pi_{h+j}\, v^j {}_jp_{x+h}. \tag{8.2.4} $$

Thus, ${}_hV$ as defined by (8.2.2) converts readily to the prospective formula: the actuarial present value of future benefits less the actuarial present value of future benefit premiums.

In Chapter 7 we discussed four types of formulas for the benefit reserve: prospective, retrospective, premium-difference, and paid-up insurance. These were applicable to benefit reserves for contracts with level benefit premiums and level benefits. Only the prospective and retrospective forms extend naturally to the general fully discrete insurance. The retrospective formula will be developed in the next section.

Example 8.2.1

A fully discrete whole life insurance with a unit benefit issued to (x) has its first year's benefit premium equal to the actuarial present value of the first year's benefit, and the remaining benefit premiums are level and determined by the equivalence principle. Determine formulas for (a) the first year's benefit premium, (b) the level benefit premium after the first year, and (c) the benefit reserve at the first duration.

Solution:

a. From Chapter 4, $\pi_0 = A^1_{x:\overline{1}|}$.
b. By the equivalence principle $A^1_{x:\overline{1}|} + \pi a_x = A_x$, so $\pi = (A_x - A^1_{x:\overline{1}|})/a_x = A_{x+1}/\ddot{a}_{x+1} = P_{x+1}$.
c. By the prospective formula, ${}_1V = A_{x+1} - \pi\ddot{a}_{x+1} = 0$. ▼

Example 8.2.1 illustrates one approach to nonlevel premiums. Another approach would be to set a premium pattern in the loss variable ${}_0L$ as defined in (8.2.1) by a set of weights, w_j, for $j = 0, 1, 2, \ldots$. Applying the equivalence principle, we have in a special case of (8.2.1)

$$ \mathrm{E}[{}_0L] = 0, $$

or

$$ \sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j} = \pi \sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x \tag{8.2.5} $$

and

$$ \pi = \frac{\sum_{j=0}^{\infty} b_{j+1}\, v^{j+1}\, {}_jp_x\, q_{x+j}}{\sum_{j=0}^{\infty} w_j\, v^j\, {}_jp_x}. \tag{8.2.6} $$

By different selections of sequences $\{b_{j+1};\ j = 0, 1, 2, \ldots\}$ and $\{w_j;\ j = 0, 1, 2, \ldots\}$, the various benefit premium formulas can be obtained.

If we consider the sequence $\{b_{j+1};\ j = 0, 1, 2, \ldots\}$ to be fixed, there remains great flexibility in selecting the sequence $\{w_j;\ j = 0, 1, 2, \ldots\}$, which in turn determines the sequence $\{\pi_j;\ j = 0, 1, 2, \ldots\}$. There may be commercial considerations to require that $w_j \geq 0$ for all j, but the equivalence principle does not impose this condition. In Example 8.2.1, $b_{j+1} = 1$ for all j, but $\pi w_0 = A^1_{x:\overline{1}|}$ and $\pi w_j = (A_x - A^1_{x:\overline{1}|}) / a_x$, $j = 1, 2, \ldots$, and (8.2.6) is satisfied. A different application is found in Example 8.2.2.

Example 8.2.2

The annual benefit premiums for a fully discrete whole life insurance with a unit benefit issued to (x) are $\pi_j = \pi w_j$, where $w_j = (1 + r)^j$. The rate r might be selected to estimate the expected growth rate in the insured's income.

Develop formulas for

a. π
b. ${}_hV$ and
c. ${}_hV$ when $r = i$.

Solution:

a. Using (8.2.5), $\pi = A_x / \ddot{a}^*_x$, where $\ddot{a}^*_x$ is valued at the rate of interest $i^* = (i - r) / (1 + r)$. When $r = i$, $\pi = A_x / (e_x + 1)$.

b. Using (8.2.4),

$$
\begin{aligned}
{}_hV &= A_{x+h} - \sum_{j=0}^{\infty} \pi_{j+h}\, v^j\, {}_jp_{x+h} \\
&= A_{x+h} - \frac{A_x}{\ddot{a}^*_x} (1 + r)^h \sum_{j=0}^{\infty} \left(\frac{1 + r}{1 + i}\right)^j {}_jp_{x+h} \\
&= A_{x+h} - \frac{A_x}{\ddot{a}^*_x} (1 + r)^h \ddot{a}^*_{x+h}.
\end{aligned}
$$

c. ${}_hV = A_{x+h} - [A_x / (e_x + 1)](1 + r)^h (e_{x+h} + 1)$. ▼

In Example 8.2.2, negative benefit reserves are possible with higher values of r. See Exercise 8.32 for a variation of this policy.

Now consider a general fully continuous insurance on (x) under which

- The death benefit payable at the moment of death, t, is b_t, and
- Benefit premiums are payable continuously at t at the annual rate, π_t.

The prospective loss for a life insured at x and surviving at t is the present value at t of the future benefits less the present value at t of the future benefit premiums:

$$
{}_tL = \begin{cases} 0 & T(x) \le t \\ b_{T(x)}\, v^{T(x)-t} - \displaystyle\int_t^{T(x)} \pi_u\, v^{u-t}\, du & T(x) > t. \end{cases} \tag{8.2.7}
$$

The benefit reserve for this general case, which we will denote by ${}_t\bar{V}$, is then

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}[{}_tL | T(x) > t] \\
&= \mathrm{E}\left[b_{T(x)}\, v^{T(x)-t} - \int_t^{T(x)} \pi_u\, v^{u-t}\, du | T(x) > t \right] \\
&= \mathrm{E}\left[b_{(T(x)-t)+t}\, v^{T(x)-t} - \int_0^{T(x)-t} \pi_{t+r}\, v^r\, dr | T(x) > t \right].
\end{aligned} \tag{8.2.8}
$$

As assumed to obtain (8.2.3) for the fully discrete insurance, we assume here that the conditional distribution of $T(x) - t$, given $T(x) > t$, is the same as the distribution of $T(x + t)$ and proceed to

$$
\begin{aligned}
{}_t\bar{V} &= \mathrm{E}\left[b_{T(x+t)+t}\, v^{T(x+t)} - \int_0^{T(x+t)} \pi_{t+r}\, v^r\, dr \right] \\
&= \int_0^\infty \left(b_{t+u}\, v^u - \int_0^u \pi_{t+r}\, v^r\, dr \right) {}_up_{x+t}\, \mu_x(t+u)\, du \\
&= \int_0^\infty b_{t+u}\, v^u\, {}_up_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+r}\, v^r\, {}_rp_{x+t}\, dr.
\end{aligned} \tag{8.2.9}
$$

The second integral in (8.2.9) is obtained by integration by parts, or, alternatively, by reversing the order of integration. In other words, ${}_t\bar{V}$ can be expressed as the actuarial present value of future benefits less the actuarial present value of future benefit premiums. If the assumption about the conditional distribution of $T(x) - t$, given $T(x) > t$, does not hold, we are in the select mortality mode.

8.3 Recursion Relations for Fully Discrete Benefit Reserves

One objective of this chapter is to explore recursion relations among the loss random variables, their expected values, and variances. We start with a definition for the insurer's net cash loss (negative cash flow) within each insurance year for the fully discrete model as defined for (8.2.1). Figure 8.3.1 is a time diagram that shows the annual cash income and cash outgo.

Insurer's Cash Income and Outgo for General Fully Discrete Insurance

Outgo		0	0	0		0	0	$b_{K(x)+1}$	0	0	
Income	π_0	π_1	π_2	π_3		$\pi_{K(x)-1}$	$\pi_{K(x)}$	0	0	0	etc.
	0	1	2	3		$K(x)-1$	$K(x)$	$K(x)+1$	$K(x)+2$	$K(x)+3$	

Let C_h denote the present value at h of the net cash loss during the year $(h, h+1)$. If $(h, h+1)$ is before the year of death $[h < K(x)]$, then $C_h = -\pi_h$. If $(h, h+1)$ is the year of death $[h = K(x)]$, then $C_h = v\, b_{h+1} - \pi_h$. And if $(h, h+1)$ is after the year of death, of course, $C_h = 0$. Restating this definition as an explicit function of $K(x)$,

$$C_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ v\, b_{h+1} - \pi_h & K(x) = h \\ -\pi_h & K(x) = h+1, h+2, \ldots . \end{cases} \tag{8.3.1}$$

For the conditional distribution of C_h, given $K(x) > h$, we observe that

$$C_h = v b_{h+1}\, I - \pi_h,$$

where

$$I = \begin{cases} 1 & \text{with probability } q_{x+h} \\ 0 & \text{with probability } p_{x+h}. \end{cases}$$

Therefore,

$$\mathrm{E}[C_h | K(x) \geq h] = v\, b_{h+1}\, q_{x+h} - \pi_h \tag{8.3.2}$$

and

$$\mathrm{Var}[C_h | K(x) \geq h] = (v\, b_{h+1})^2\, q_{x+h}\, p_{x+h}. \tag{8.3.3}$$

Moreover, using (2.2.10) and (2.2.11) along with (8.3.1)–(8.3.3),

$$\mathrm{E}[C_h] = (v\, b_{h+1}\, q_{x+h} - \pi_h)\, {}_hp_x \tag{8.3.4}$$

and

$$\mathrm{Var}(C_h) = (v\, b_{h+1}\, q_{x+h} - \pi_h)^2\, {}_hp_x\, {}_hq_x + (v\, b_{h+1})^2 q_{x+h}\, p_{x+h}\, {}_hp_x. \tag{8.3.5}$$

Finally, for $j > h$, C_j and C_h are correlated, an assertion that is left for the student to verify in Exercises 8.5 and 8.6.

As previously defined in (8.2.1), ${}_hL$ is the present value at h of the insurer's future cash outflow less the present value at h of the insurer's future cash income. By rearranging the terms in this definition, an equivalent one that states ${}_hL$ as the sum of the present values at h of the insurer's future net annual cash losses is obtained. This is

$${}_hL = \sum_{j=h}^{\infty} v^{j-h}\, C_j. \tag{8.3.6}$$

For $h < K(x)$, we have from (8.3.1),

$$\begin{aligned} {}_hL &= \sum_{j=h}^{K(x)} v^{j-h}\, C_j = v^{K(x)-h}\, (v\, b_{K(x)+1} - \pi_{K(x)}) - \sum_{j=h}^{K(x)-1} v^{j-h} \pi_j \\ &= v^{K(x)+1-h}\, b_{K(x)+1} - \sum_{j-h}^{K(x)} v^{j-h}\, \pi_j \end{aligned}$$

as before. For $h = K(x)$, ${}_hL = C_{K(x)} = (v\, b_{K(x)+1} - \pi_{K(x)})$. And for $h > K(x)$, both sides of (8.3.6) are zero.

A recursion relation for the loss variables follows from (8.3.6):

$$ {}_hL = C_h + v \sum_{j=h+1}^{\infty} v^{j-(h+1)} C_j = C_h + v\ {}_{h+1}L. \tag{8.3.7} $$

A recursion relation for the benefit reserves can be obtained from (8.3.7) by

$$ \begin{aligned} {}_hV &= \mathrm{E}[{}_hL | K(x) \geq h] \\ &= \mathrm{E}[C_h + v\ {}_{h+1}L | K(x) \geq h] \\ &= v\ b_{h+1}\ q_{x+h} - \pi_h + v\ \mathrm{E}[{}_{h+1}L | K(x) \geq h]. \end{aligned} \tag{8.3.8} $$

Since ${}_{h+1}L$ is zero when $K(x)$ is h, we have

$$ \begin{aligned} {}_hV &= v\ b_{h+1}\ q_{x+h} - \pi_h + v\ \mathrm{E}[{}_{h+1}L | K(x) \geq h+1]\ p_{x+h} \\ &= v\ b_{h+1}\ q_{x+h} - \pi_h + v\ {}_{h+1}V\ p_{x+h}. \end{aligned} \tag{8.3.9} $$

Formula (8.3.9) is a backward recursion formula [$u(h) = c(h) + d(h) \times u(h+1)$] for the general fully discrete benefit reserve. Note that $d(h) = vp_{x+h}$ again and $c(h) = vb_{h+1}q_{x+h} - \pi_h$. A forward recursion formula can be obtained by solving (8.3.9) for ${}_{h+1}V$. (See Exercise 8.2.) This forward formula was used in Examples 7.4.3 and 7.4.4 in an aggregate mode; that is, the mortality functions were in life table form.

Further insight to the progress of benefit reserves can be gained by rearrangements of (8.3.9). First, add π_h to both sides, to see

$$ {}_hV + \pi_h = b_{h+1}\ v\ q_{x+h} + {}_{h+1}V\ v\ p_{x+h}. \tag{8.3.10} $$

In words, the resources required at the beginning of insurance year $h+1$ equal the actuarial present value of the year-end requirements. The sum ${}_hV + \pi_h$ is called the ***initial benefit reserve*** for the policy year $h+1$. In contrast, ${}_hV$ and ${}_{h+1}V$ are called the ***terminal benefit reserves*** for insurance years h and $h+1$ to indicate that they are year-end benefit reserves.

Formula (8.3.10) can be rearranged to separate the benefit premium π_h into components for insurance year $h+1$, namely,

$$ \pi_h = b_{h+1}\ v\ q_{x+h} + ({}_{h+1}V\ v\ p_{x+h} - {}_hV). \tag{8.3.11} $$

The first component on the right-hand side of (8.3.11) is the 1-year term insurance benefit premium for the sum insured b_{h+1}. The second component, ${}_{h+1}V\ v\ p_{x+h} - {}_hV$, represents the amount which, if added to ${}_hV$ at the beginning of the year, would accumulate under interest and survivorship to ${}_{h+1}V$ at the end of the year.

For the purpose of subsequent comparison with formulas for a fully continuous insurance, we multiply both sides of (8.3.11) by $1+i$ and rearrange the formula to

$$ \pi_h + ({}_hV + \pi_h)i + {}_{h+1}Vq_{x+h} = b_{h+1}\ q_{x+h} + \Delta({}_hV). \tag{8.3.12} $$

The left-hand side of (8.3.12) indicates resources for insurance year $h + 1$, namely, the benefit premium, interest for the year on the initial benefit reserve, and the expected release by death of the terminal benefit reserve. The right-hand side consists of the expected payment of the death benefit at the end of the year and the increment ${}_{h+1}V - {}_{h}V$ in the benefit reserve.

An analysis different than (8.3.10)–(8.3.12) results if one considers that the benefit reserve ${}_{h+1}V$ is to be available to offset the death benefit b_{h+1}, and that only the ***net amount at risk***, $b_{h+1} - {}_{h+1}V$, needs to be covered by 1-year term insurance. For this analysis we have, on substituting $1 - q_{x+h}$ for p_{x+h} in (8.3.10) and multiplying through by $1 + i$,

$$ {}_{h+1}V = ({}_{h}V + \pi_h)(1 + i) - (b_{h+1} - {}_{h+1}V)q_{x+h}. \tag{8.3.13} $$

Corresponding to (8.3.11), we now have

$$ \pi_h = (b_{h+1} - {}_{h+1}V)v\, q_{x+h} + (v\, {}_{h+1}V - {}_{h}V). \tag{8.3.14} $$

The first component on the right-hand side of (8.3.14) is the 1-year term insurance benefit premium for the net amount of risk. The second component, $v\, {}_{h+1}V - {}_{h}V$, is the amount which, if added to ${}_{h}V$ at the beginning of the year, would accumulate under interest to ${}_{h+1}V$ at the end of the year. In this formulation ${}_{h+1}V$ is used, in case of death, to offset the death benefit. Consequently, the benefit reserve accumulates as a savings fund. This is shown again by the formula corresponding to (8.3.12), namely,

$$ \pi_h + ({}_{h}V + \pi_h)i = (b_{h+1} - {}_{h+1}V)q_{x+h} + \Delta({}_{h}V), \tag{8.3.15} $$

which is left for the reader to interpret.

The analysis by (8.3.11) does not use the benefit reserve to offset the death benefit, and consequently the benefit reserve accumulates under interest and survivorship. Both components of the right-hand side of (8.3.11) involve mortality risk, whereas in (8.3.14) only the first component does. We see in Section 8.5 that (8.3.14) is related to a flexible means for calculating the variance of loss attributable to the random nature of time until death.

Formulas (8.3.10)–(8.3.15) are all recursion relations for the benefit reserve at integral durations. None of the six is written in the form of an explicit backward or forward formula; rather, each is written to give an insight. In Example 8.3.1, recursion relation (8.3.14) is used to obtain an explicit formula for the benefit premium and benefit reserve.

Example 8.3.1

A deferred whole life annuity-due issued to (x) for an annual income of 1 commencing at age $x + n$ is to be paid for by level annual benefit premiums during the deferral period. The benefit for death prior to age $x + n$ is the benefit reserve. Assuming the death benefit is paid at the end of the year of death, determine the annual benefit premium and the benefit reserve at the end of year k for $k \leq n$.

Solution:

Using the fact that $b_{h+1} = {}_{h+1}V$ for $h = 0, 1, 2, \ldots, n-1$, in (8.3.14), we have

$$\pi = v\ {}_{h+1}V - {}_{h}V.$$

On multiplication by v^h, we have

$$\pi v^h = v^{h+1}\ {}_{h+1}V - v^h\ {}_{h}V = \Delta(v^h\ {}_{h}V). \tag{8.3.16}$$

Summing over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\ {}_{n}V - v^0\ {}_{0}V = \pi \sum_{h=0}^{n-1} v^h = \pi \ddot{a}_{\overline{n}|},$$

and, since ${}_{0}V = 0$ and ${}_{n}V = \ddot{a}_{x+n}$, it follows that

$$\pi = v^n \frac{\ddot{a}_{x+n}}{\ddot{a}_{\overline{n}|}} = \frac{\ddot{a}_{x+n}}{\ddot{s}_{\overline{n}|}}.$$

Thus, this annuity is identical to that described in Example 6.6.2. The benefit reserve at the end of k years can be found by summing (8.3.16) over $h = 0, 1, 2, \ldots, k-1$ to give

$$v^k\ {}_{k}V = \pi \ddot{a}_{\overline{k}|},$$

from which

$${}_{k}V = \pi \ddot{s}_{\overline{k}|}. \qquad \blacktriangledown$$

Example 8.3.2

A fully discrete n-year endowment insurance on (x) provides, in case of death within n years, a payment of 1 plus the benefit reserve. Obtain formulas for the level benefit premium and the benefit reserve at the end of k years, given that the maturity value is 1.

Solution:

In this case $b_h = 1 + {}_{h}V$, and the net amount at risk has constant value 1. Denoting the annual benefit premium by π and using (8.3.14), we have

$$v\ {}_{h+1}V - {}_{h}V = \pi - v\ q_{x+h} \qquad h = 0, 1, \ldots, n-1.$$

On multiplication by v^h, this becomes

$$\Delta(v^h\ {}_{h}V) = \pi\ v^h - v^{h+1}\ q_{x+h}. \tag{8.3.17}$$

Summing this over $h = 0, 1, 2, \ldots, n-1$, we obtain

$$v^n\ {}_{n}V = \pi\ \ddot{a}_{\overline{n}|} - \sum_{h=0}^{n-1} v^{h+1} q_{x+h}$$

so that, with ${}_{n}V = 1$ (the maturity value is 1),

$$\pi = \frac{v^n + \sum_{h=0}^{n-1} v^{h+1} q_{x+h}}{\ddot{a}_{\overline{n}|}}.$$

By summing (8.3.17) over $h = 0, 1, 2, \ldots, k-1$, and solving for ${}_kV$, we have

$${}_kV = \pi \ddot{s}_{\overline{k}|} - \sum_{h=0}^{k-1} (1+i)^{k-h-1} q_{x+h}.$$

▼

Just before Example 8.2.1 we promised to develop a retrospective formula for the benefit reserve of the general fully discrete insurance in this section. We start by rewriting recursion relation (8.3.11) in the form

$$\pi_h - b_{h+1}\, v\, q_{x+h} = {}_{h+1}V\, v\, p_{x+h} - {}_hV$$

and then multiplying both sides by $v^h\, {}_hp_x$ to obtain

$$\begin{aligned} \pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h} &= {}_{h+1}V\, v^{h+1}\, {}_{h+1}p_x - {}_hV\, v^h\, {}_hp_x \\ &= \Delta({}_hV\, v^h\, {}_hp_x), \end{aligned} \tag{8.3.18}$$

which holds for $h = 0, 1, 2, \ldots$. When we sum both sides of (8.3.18) over the values from 0 to $k-1$, we have

$$\sum_{h=0}^{k-1} (\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}) = {}_kV\, v^k\, {}_kp_x - {}_0V.$$

With equivalence principle premiums, ${}_0V = 0$, so we can rewrite this last equation for the terminal benefit reserve of the general fully discrete insurance as

$${}_kV = \sum_{h=0}^{k-1} \frac{\pi_h\, v^h\, {}_hp_x - b_{h+1}\, v^{h+1}\, {}_hp_x\, q_{x+h}}{v^k\, {}_kp_x}$$

and then as

$${}_kV = \sum_{h=0}^{k-1} (\pi_h - v b_{h+1}\, q_{x+h}) \frac{(1+i)^{k-h}}{{}_{k-h}p_{x+h}}. \tag{8.3.19}$$

Formula (8.3.19) shows the benefit reserve at k as the sum over the first k years of each year's premium less its expected death benefit accumulated with respect to interest and mortality to k.

8.4 Benefit Reserves at Fractional Durations

We consider again the general fully discrete insurance of (8.2.1) on (x) for a death benefit of b_{j+1} at the end of insurance year $j+1$, purchased by annual benefit premiums of π_j, $j = 0, 1, \ldots$, payable at the beginning of the insurance year. We seek a formula for, and an approximation to, the ***interim benefit reserve,*** that is, ${}_{h+s}V$ for $h = 0, 1, 2, \ldots$ and $0 < s < 1$. Extending the earlier definition of the benefit reserve as stated in (8.2.1) and (8.2.2), we have for the interim case

$${}_{h+s}L = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ v^{1-s}\, b_{K(x)+1} & K(x) = h \\ v^{K(x)+1-(h+s)}\, b_{K(x)+1} - \displaystyle\sum_{j=h+1}^{K(x)} v^{j-(h+s)}\, \pi_j & K(x) = h+1, h+2, \ldots \end{cases}$$

(8.4.1)

and

$$ {}_{h+s}V = \mathrm{E}[{}_{h+s}L|T(x) > h + s]. \tag{8.4.2} $$

From (8.4.2),

$$ {}_{h+s}V = v^{1-s}\, b_{h+1}\, {}_{1-s}q_{x+h+s} + v^{1-s}\, {}_{h+1}V\, {}_{1-s}p_{x+h+s}. \tag{8.4.3} $$

Now multiply both sides of (8.4.3) by $v^s\, {}_sp_{x+h}$ to obtain

$$ v^s\, {}_sp_{x+h}\, {}_{h+s}V = v\, b_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V)\, p_{x+h}. \tag{8.4.4} $$

Equation (8.3.9) provides an expression for $vb_{h+1}\, q_{x+h}$, which can be substituted into (8.4.4) to obtain

$$ v^s\, {}_sp_{x+h}\, {}_{h+s}V = ({}_hV + \pi_h - {}_{h+1}V\, v\, p_{x+h})\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + v\, {}_{h+1}V\, p_{x+h}, $$

which can be rearranged to

$$ v^s\, {}_sp_{x+h}\, {}_{h+s}V = ({}_hV + \pi_h)\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + ({}_{h+1}V\, v\, p_{x+h})\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.5} $$

This exact expression shows that when the interim benefit reserve at $h + s$ is discounted with respect to interest and mortality to h, the result is equal to an interpolated value between the initial benefit reserve at h and the value of the terminal benefit reserve at $h + 1$ discounted to h.

We emphasize that the interpolation is, in general, not linear; however, under the assumption of uniform distribution of deaths over the age interval the interpolation weights are linear and (8.4.5) is

$$ v^s\, {}_sp_{x+h}\, {}_{h+s}V = ({}_hV + \pi_h)(1 - s) + ({}_{h+1}V\, v\, p_{x+h})(s). \tag{8.4.6} $$

By replacing i and q_{x+h} with zeros in (8.4.6), as an approximation, the result is linear interpolation between the initial benefit reserve at h and the terminal benefit reserve. The approximate result is

$$ {}_{h+s}V = (1 - s)({}_hV + \pi_h) + s({}_{h+1}V), \tag{8.4.7} $$

which is often written in the form

$$ {}_{h+s}V = (1 - s)({}_hV) + s({}_{h+1}V) + (1 - s)\pi_h. \tag{8.4.8} $$

Here the interim benefit reserve is the sum of the value obtained by linear interpolation between the terminal benefit reserves,

$$ (1 - s)({}_hV) + s({}_{h+1}V), $$

and the ***unearned benefit premium*** $(1 - s)\pi_h$. In general,

(the unearned benefit premium at a given time during the year) = (the benefit premium for the year) × (the difference between the time through which the premium has been paid and the given time).

Thus, on an annual premium basis, the benefit premium has been paid to the end of the year so at time s the unearned benefit premium is $(1 - s)\pi_h$. This notion of an unearned benefit premium will be used in discussing approximations to benefit reserves when the premiums are collected by installments more frequent than an annual basis.

We consider now one such case, that of true semiannual premiums with claims paid at the end of the insurance year of death. For $0 < s \leq 1/2$ we could start with the random variable giving the present value of prospective losses as of time $h + s$ and then calculate its conditional expectation given that (x) has survived to $h + s$. This is a bit more complex than it was for (8.4.4), so we start with the equation corresponding to (8.4.3) by noting that it is the prospective formula

$$ {}_{h+s}V^{(2)} = v^{1-s}\, b_{h+1}({}_{1-s}q_{x+h+s}) + v^{1-s}({}_{h+1}V^{(2)})\ {}_{1-s}p_{x+h+s} $$

$$ - \frac{\pi_h}{2}\,(v^{0.5-s})({}_{0.5-s}p_{x+h+s}). \tag{8.4.9} $$

The first two terms of (8.4.9) can be viewed as the actuarial present value of the death benefit and an endowment benefit of amount equal to the reserve, and the third term is the actuarial present value of the future benefit premium for the $1 - s$ year endowment insurance. Multiplying both sides of (8.4.9) by ${}_sp_{x+h}\, v^s$, we have

$$ {}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} = vb_{h+1}({}_{s|1-s}q_{x+h}) + v({}_{h+1}V^{(2)})\ p_{x+h} $$

$$ - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h}). \tag{8.4.10} $$

For the semiannual premium policy the equation corresponding to (8.3.10) is also a prospective benefit reserve formula:

$$ {}_hV^{(2)} = b_{h+1}vq_{x+h} + {}_{h+1}V^{(2)}vp_{x+h} - \frac{\pi_h}{2}\,(1 + v^{0.5}{}_{0.5}p_{x+h}). \tag{8.4.11} $$

Equation (8.4.11) provides an expression for vb_{h+1} to substitute in (8.4.10), which yields

$$ {}_sp_{x+h}\, v^s\ {}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} $$

$$ + \left[v({}_{h+1}V^{(2)})\ p_{x+h} - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h})\right] $$

$$ \times \left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.12} $$

Formula (8.4.12) corresponds to (8.4.5), which showed that the interim benefit reserve at $h + s$, discounted with respect to interest and mortality to h, is equal to a nonlinear interpolated value between the initial benefit reserve at h and the discounted value to h of the terminal benefit reserve at $h + 1$. For the semiannual premium case, this terminal reserve has been reduced by the amount of the

discounted value of the midyear benefit premium. Under the assumption of a uniform distribution of deaths over the age interval, we have linear interpolation on the right-hand side:

$$ {}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)(1 - s) + \left[v({}_{h+1}V^{(2)})\, p_{x+h} - \frac{\pi_h}{2}\,(v^{0.5})({}_{0.5}p_{x+h})\right](s). \tag{8.4.13} $$

Again setting i and q_{x+h} equal to zero, as an approximation, we obtain simple linear interpolation between the initial benefit reserve and the terminal benefit reserve reduced by the benefit premium due at midyear:

$$ {}_{h+s}V^{(2)} = \left({}_hV^{(2)} + \frac{\pi_h}{2}\right)(1 - s) + \left({}_{h+1}V^{(2)} - \frac{\pi_h}{2}\right)(s). $$

This formula can be rearranged as the interpolated value between the terminal benefit reserves plus the unearned benefit premium $(\pi_h)(1/2 - s)$:

$$ {}_{h+s}V^{(2)} = [(1 - s)\,{}_hV^{(2)} + s\,{}_{h+1}V^{(2)}] + \left(\frac{1}{2} - s\right)\pi_h. \tag{8.4.14} $$

For the last half of the year when $1/2 < s \leq 1$, we can proceed as above to obtain the following exact formula for the benefit reserve at $h + s$ discounted with respect to interest and mortality to h:

$$ {}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = \left[{}_hV^{(2)} + \frac{\pi_h}{2}\,(1 + v^{0.5}\,{}_{0.5}p_{x+h})\right]\frac{{}_{s|1-s}q_{x+h}}{q_{x+h}} + [v({}_{h+1}V^{(2)})\, p_{x+h}]\left(1 - \frac{{}_{s|1-s}q_{x+h}}{q_{x+h}}\right). \tag{8.4.15} $$

Again under uniform distribution of death in the year of age, we have the linear interpolation

$$ {}_sp_{x+h}\, v^s\, {}_{h+s}V^{(2)} = (1 - s)\left[{}_hV^{(2)} + \frac{\pi_h}{2}\,(1 + v^{0.5}\,{}_{0.5}p_{x+h})\right] + s[v({}_{h+1}V^{(2)})\, p_{x+h}], \tag{8.4.16} $$

and with i and q_{x+h} set equal to zero, as an approximation, we have the simple linear interpolation plus the unearned benefit premium

$$ {}_{h+s}V^{(2)} = {}_hV^{(2)}(1 - s) + {}_{h+1}V^{(2)}(s) + \pi_h(1 - s). \tag{8.4.17} $$

(For the general result for m-thly reserves see Exercise 8.12.)

8.5 Allocation of the Risk to Insurance Years

In Section 8.3 recursion relations for benefit reserves are developed by an analysis of the insurer's annual cash income and cash outflow. Now we extend this analysis to an accrual or incurred basis and develop allocations of the risk, as measured by

the variance of the loss variables, to the insurance years. Figure 8.5.1 shows in a time diagram the insurer's annual cash incomes, cash outflows, and changes in liability for the general fully discrete insurance of (8.2.1). The random variable C_h is related to the cash flows of the policy year $(h, h + 1)$. We now define a random variable related to the total change in liability, cash flow, and reserves.

Insurer's Cash Incomes, Outflows, and Changes in Liability for Fully Discrete General Insurance

Outflow		0	0	0	$b_{K(x)+1}$	0	
Income	π_0	π_1	π_2	$\pi_{K(x)}$	0	0	
ΔLiability		${}_1V$	${}_2V-(1+i)\,{}_1V$	${}_{K(x)}V-(1+i){}_{K(x)-1}V$	$-(1+i){}_{K(x)}V$	0	etc.
	0	1	2	$K(x)$	$K(x)+1$	$K(x)+2$	

Let Λ_h denote the present value at h (a non-negative integer) of the insurer's cash loss plus change in liability during the year $(h, h + 1)$. If $(h, h + 1)$ is before the year of death $[h < K(x)]$, then

$$\Lambda_h = C_h + v\ \Delta\text{Liability} = -\pi_h + v\ {}_{h+1}V - {}_hV.$$

If $(h, h + 1)$ is the year of death $[h = K(x)]$, then

$$\Lambda_h = C_h + v\Delta\text{Liability} = v\ b_{h+1} - \pi_h - {}_hV.$$

And if $(h, h + 1)$ is after the year of death, of course $\Lambda_h = 0$. Restating this definition as a function of $K(x)$, and rearranging the terms,

$$\Lambda_h = \begin{cases} 0 & K(x) = 0, 1, \ldots, h-1 \\ (v\ b_{h+1} - \pi_h) + (-{}_hV) & K(x) = h \\ (-\pi_h) + (v\ {}_{h+1}V - {}_hV) & K(x) = h+1, h+2, \ldots . \end{cases} \tag{8.5.1}$$

The definition of Λ_h in (8.5.1) can be rewritten to display Λ_h as the loss variable for a 1-year term insurance with a benefit equal to the amount at risk on the basic policy. See Exercise 8.31.

It follows that

$$\mathrm{E}[\Lambda_h | K(x) \geq h] = v\ b_{h+1}\ q_{x+h} + v\ {}_{h+1}V\ p_{x+h} - (\pi_h + {}_hV), \tag{8.5.2}$$

which is zero by (8.3.10).

Since the conditional distribution of Λ_h, given $K(x) = h, h + 1, \ldots$, is a two-point distribution, then

$$\mathrm{Var}[\Lambda_h | K(x) \geq h] = [v(b_{h+1} - {}_{h+1}V)]^2\ p_{x+h}\ q_{x+h}. \tag{8.5.3}$$

With $j \leq h$ we can use (2.2.10) and (2.2.11) to obtain

$$\mathrm{E}[\Lambda_h | K(x) \geq j] = 0 \tag{8.5.4}$$

and

$$\text{Var}[\Lambda_h|K(x) \geq j] = \text{Var}[\Lambda_h|K(x) \geq h]\ {}_{h-j}p_{x+j}. \tag{8.5.5}$$

Unlike the C_h's of Section 8.3, the Λ_h's are uncorrelated, an assertion that is proved in the following lemma. This fact conveys some sense of the role of reserves in stabilizing financial reporting of insurance operations.

Lemma 8.5.1:

For non-negative integers satisfying $g \leq h < j$,

$$\text{Cov}[\Lambda_h, \Lambda_j|K(x) \geq g] = 0. \tag{8.5.6}$$

Proof:

From (8.5.4), $\text{E}[\Lambda_h|K(x) \geq g] = 0$; therefore,

$$\text{Cov}[\Lambda_h, \Lambda_j|K(x) \geq g] = \text{E}[\Lambda_h\Lambda_j|K(x) \geq g].$$

From (8.5.1) we see that Λ_h is equal to the constant $(v\ {}_{h+1}V - {}_hV - \pi_h)$ where Λ_j is nonzero. Thus,

$$\Lambda_h\Lambda_j = (v\ {}_{k+1}V - {}_hV - \pi_h)\Lambda_j \qquad \text{for all } K(x), \tag{8.5.7}$$

$$\text{E}[\Lambda_h\Lambda_j|K(x) \geq g] = (v\ {}_{h+1}V - {}_hV - \pi_h)\text{E}[\Lambda_j|K(x) \geq g] = 0,$$

and

$$\text{Cov}[\Lambda_k\Lambda_j|K(x) \geq g] = 0. \qquad \blacksquare$$

We now express the loss variables ${}_hL$ in terms of the Λ_h's. From the definition of the Λ_h's and formula (8.3.6),

$$\sum_{j=h}^{\infty} v^{j-h}\,\Lambda_j = \sum_{j=h}^{\infty} v^{j-h}[C_j + v\Delta\text{Liability}\ (j, j+1)]$$

$$= {}_hL + \sum_{j=h}^{\infty} v^{j-h+1}\,\Delta\text{Liability}\ (j, j+1). \tag{8.5.8}$$

Conceptually the last term will be the present value of the final liability minus the liability at h, that is, $0 - {}_hV$. Thus, we have the relationship

$${}_hL = \sum_{j=h}^{\infty} v^{j-h}\,\Lambda_j + {}_hV. \tag{8.5.9}$$

The following recursion relations for the ${}_hL$'s follow from the definition of the Λ_h's and (8.3.7):

$${}_hL = C_h + v\ {}_{h+1}L = [\Lambda_h - (v\ {}_{h+1}V - {}_hV)] + v\ {}_{h+1}L$$

$$= \Lambda_h + v\ {}_{h+1}L + ({}_hV - v\ {}_{h+1}V). \tag{8.5.10}$$

Relationship (8.5.10) can be extended by iteration or by direct use of (8.5.9) to obtain, for $h < j$,

$${}_hL = \sum_{i=h}^{h+j-1} v^{i-h}\,\Lambda_i + v^j\ {}_{h+j}L + ({}_hV - v^j\ {}_{h+j}V). \tag{8.5.11}$$

The following result, attributed to Hattendorf, provides a means for expressing the variance of ${}_hL$ in terms of the benefit reserve values. Equally significant is the fact that the variance of ${}_hL$ can be allocated to the individual insurance years. This allocation facilitates risk management planning for a limited number of future insurance years rather than for the entire insurance period. This option permits sequential risk management decisions.

Theorem 8.5.1

$$\text{Var}[{}_hL|K(x) \geq h]$$

$$= \sum_{i=h}^{\infty} v^{2(i-h)} \text{Var}[\Lambda_i|K(x) \geq h] \tag{8.5.12}$$

$$= \text{Var}[\Lambda_h|K(x) \geq h] + v^2 \text{Var}[{}_{h+1}L|K(x) \geq h] \tag{8.5.13}$$

$$= \sum_{i=h}^{h+j-1} v^{2(i-h)} \text{Var}[\Lambda_i|K(x) \geq h] + v^{2j} \text{Var}[{}_{h+j}L|K(x) \geq h]. \tag{8.5.14}$$

Proof:

(a) follows from (8.5.9) and Lemma 8.5.1.
(b) is a special case ($j = 1$) of (c) which follows from (8.5.11) and Lemma 8.5.1. ■

Theorem 8.5.2

$$\text{Var}[{}_{h+j}L|K(x) \geq h] = {}_jp_{x+h} \text{Var}[{}_{h+j}L|K(x) \geq h + j] \tag{8.5.15}$$

Proof:

By (8.5.9),

$$\text{Var}[{}_{h+j}L|K(x) \geq h] = \text{Var}\left[\sum_{i=h+j}^{\infty} v^{i-(h+j)} \Lambda_i + {}_{h+j}V|K(x) \geq h\right]$$

$$= \sum_{i=h+j}^{\infty} v^{2[i-(h+j)]} \text{Var}[\Lambda_i|K(x) \geq h],$$

which, from (8.5.5),

$$= \sum_{i=h+j}^{\infty} v^{2[i-(h+j)]} {}_jp_{x+h} \text{Var}[\Lambda_i|K(x) \geq h + j]$$

$$= {}_jp_{x+h} \text{Var}[{}_{h+j}L|K(x) \geq h + j]. \quad ■$$

Combining the results of these two theorems, we have the following corollary.

Corollary:

$$\text{Var}[{}_hL|K(x) \geq h]$$

$$\text{a.} = \sum_{j=h}^{\infty} v^{2(j-h)}\, {}_{j-h}p_{x+h}\, \mathrm{Var}[\Lambda_j | K(x) \geq j] \tag{8.5.16}$$

$$\text{b.} = \mathrm{Var}[\Lambda_h | K(x) \geq h] + v^2\, p_{x+h} \mathrm{Var}[{}_{h+1}L | K(x) \geq h+1] \tag{8.5.17}$$

$$\text{c.} = \sum_{i=h}^{h+j-1} v^{2(i-h)}\, {}_{i-h}p_{x+h}\, \mathrm{Var}[\Lambda_i | K(x) \geq i] + v^{2j}\, {}_{j}p_{x+h}\, \mathrm{Var}[{}_{h+j}L | K(x) \geq h+j]. \tag{8.5.18}$$

Proof:

- For (a), apply (8.5.5) to each term of (8.5.12).
- For (b), apply Theorem 8.5.2 to (8.5.13).
- For (c), apply Theorem 8.5.2 to (8.5.14). ■

We refer to these theorems and their corollary as the Hattendorf theorem, and we illustrate their application in the following two examples. Items (b) and (c) of the corollary can be used as backward recursion formulas that are useful for understanding the duration allocation of risk and, perhaps, for computing.

Example 8.5.1

Consider an insured from Example 7.4.3 who has survived to the end of the second policy year. For this insured, evaluate

a. $\mathrm{Var}[{}_2L | K(50) \geq 2]$ directly
b. $\mathrm{Var}[{}_2L | K(50) \geq 2]$ by means of the Hattendorf theorem
c. $\mathrm{Var}[{}_3L | K(50) \geq 3]$
d. $\mathrm{Var}[{}_4L | K(50) \geq 4]$.

Solution:

a. For the direct calculation, we need a table of values for ${}_2L$.

Outcome of $K(50) - 2 = j$	${}_2L$	Conditional Probability of Outcome
0	$1{,}000v - 6.55692\,\ddot{a}_{\overline{1}\rceil} = 936.84$	${}_{0\vert}q_{52} = 0.0069724$
1	$1{,}000v^2 - 6.55692\,\ddot{a}_{\overline{2}\rceil} = 877.25$	${}_{1\vert}q_{52} = 0.0075227$
2	$1{,}000v^3 - 6.55692\,\ddot{a}_{\overline{3}\rceil} = 821.04$	${}_{2\vert}q_{52} = 0.0081170$
≥3	$0 - 6.55692\,\ddot{a}_{\overline{3}\rceil} = -18.58$	${}_3p_{52} = 0.9773879$

Then $\mathrm{E}[{}_2L | K(50) \geq 2] = 1.64$, in agreement with the value shown in Example 7.4.3 and

$$\begin{aligned}\mathrm{Var}[{}_2L | K(50) \geq 2] &= \mathrm{E}[{}_2L^2 | K(50) \geq 2] - (\mathrm{E}[{}_2L | K(50) \geq 2])^2 \\ &= 17{,}717.82 - (1.64)^2 \\ &= 17{,}715.1.\end{aligned}$$

b. To apply the Hattendorf theorem, we can use the benefit reserves from Example 7.4.3 to calculate the variances of the losses associated with the 1-year term insurances.

j	q_{52+j}	$v^2\,(1{,}000 - 1{,}000\ {}_{2+j+1}V^{1}_{50:\overline{5}\rceil})^2\ p_{52+j}\ q_{52+j}$
0	0.0069724	6 140.842
1	0.0075755	6 674.910
2	0.0082364	7 269.991

Then by (8.5.16),

$$\text{Var}[{}_2L|K(50) \geq 2] = 6{,}140.842 + (1.06)^{-2}(6{,}674.910)p_{52} + (1.06)^{-4}(7{,}269.991)_2p_{52} = 17{,}715.1,$$

which agrees with the value found by the direct calculation in part (a).

Note that in the direct method it was necessary to consider the gain in the event of survival to age 55; but for the Hattendorf theorem, we need to consider only the losses associated with the 1-year term insurances for the net amounts at risk in the remaining policy years. Thereafter, the net amount at risk is 0, and the corresponding terms in (8.5.16) vanish.

Also note that the standard deviation, $\sqrt{17{,}751.1} = 133.1$, for a single policy is more than 80 times the benefit reserve, $\text{E}[{}_2L|K(50)=2, 3, \ldots] = 1.64$.

Similarly, we use (8.5.16) to calculate

c. $\text{Var}[{}_3L|K(50) \geq 3] = 6{,}674.910 + (1.06)^{-2}(7{,}269.991)\ p_{53} = 13{,}096.2$

d. $\text{Var}[{}_4L|K(50) \geq 4] = 7{,}269.991$, or after rounding, 7,270.0. ▼

Example 8.5.2

Consider a portfolio of 1,500 policies of the type described in Example 7.4.3 and discussed in Example 8.5.1. Assume all policies have annual premiums due immediately. Further, assume 750 policies are at duration 2, 500 are at duration 3, and 250 are at duration 4, and that the policies in each group are evenly divided between those with 1,000 face amount and those with 3,000 face amount.

a. Calculate the aggregate benefit reserve.

b. Calculate the variance of the prospective losses over the remaining periods of coverage of the policies assuming such losses are independent. Also, calculate the amount which, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the future obligations to this block of business.

c. Calculate the variance of the losses associated with the 1-year term insurances for the net amounts at risk under the policies and the amount of supplement to the aggregate benefit reserve that, on the basis of the normal approximation, will give the insurer a probability of 0.95 of meeting the obligations to this block of business for the 1-year period.

d. Redo (b) and (c) with each set of policies increased 100-fold in number.

Solution:

a. Let Z be the sum of the prospective losses on the 1,500 policies. The symbols $E[Z]$ and $\text{Var}(Z)$ used below for the mean and variance of the portfolio of 1,500 policies are abridged, for in both cases the expectations are to be computed with respect to the set of conditions given above for the insureds. Using the results of Example 7.4.3, we have for the aggregate benefit reserve

$$\begin{aligned} E[Z] &= [375(1) + 375(3)](1.64) + [250(1) + 250(3)](1.73) \\ &\quad + [125(1) + 125(3)](1.21) \\ &= 4{,}795. \end{aligned}$$

b. From Example 8.5.1, we have

$$\begin{aligned} \text{Var}(Z) &= [375(1) + 375(9)](17{,}715.1) \\ &\quad + [250(1) + 250(9)](13{,}096.2) \\ &\quad + [125(1) + 125(9)](7{,}270.0) \\ &= (1.0825962) \times 10^8 \end{aligned}$$

and $\sigma_Z = 10{,}404.8$.

Then, if

$$0.05 = \Pr(Z > c) = \Pr\left(\frac{Z - 4{,}795.0}{10{,}404.8} > \frac{c - 4{,}795.0}{10{,}404.8}\right),$$

the normal approximation would imply

$$\frac{c - 4{,}795.0}{10{,}404.8} = 1.645,$$

or

$$c = 21{,}911,$$

which is 4.6 times the aggregate benefit reserve, $E[Z]$.

c. Here we take account of only the next year's risk. For each policy, we consider a variable equal to the loss associated with a 1-year term insurance for the net amount at risk. Let Z_1 be the sum of these loss variables. The expected loss for each of the 1-year term insurances is 0, hence $E[Z_1] = 0$.

From the table in part (b) of Example 8.5.1 we can obtain the variances of the losses in regard to the 1-year term insurances, and hence

$$\begin{aligned} \text{Var}(Z_1) &= [375(1) + 375(9)](6{,}140.8) + [250(1) + 250(9)](6{,}674.9) \\ &\quad + [125(1) + 125(9)](7{,}270.0) \\ &= (4.880275) \times 10^7 \end{aligned}$$

and $\sigma_{Z_1} = 6985.9$.

If c_1 is the required supplement to the aggregate benefit reserve, then

$$0.05 = \Pr(Z_1 > c_1) = \Pr\left(\frac{Z_1 - 0}{6{,}985.9} > \frac{c_1 - 0}{6{,}985.9}\right),$$

and we determine, again by the normal approximation,

$$c_1 = (1.645)(6{,}985.9) = 11{,}492,$$

which is 2.4 times the aggregate benefit reserve 4,795.

d. In this case, $E[Z] = 479{,}500$ and $\text{Var}(Z) = (1.0825962) \times 10^{10}$. By the normal approximation the amount c required to provide a probability of 0.95 that all future obligations will be met is

$$479{,}500 + 1.645\sqrt{1.0825962} \times 10^5 = 650{,}659,$$

which is 1.36 times the aggregate benefit reserve $E[Z]$.

Also, $\text{Var}(Z_1)$ is now $(4.880275) \times 10^9$. The amount c_1 of supplement to the aggregate benefit reserve required to give a 0.95 probability that the insurer can meet policy obligations for the next year is $1.645\sqrt{4.880275} \times 10^{4.5} = 114{,}918$, or 24% of the aggregate benefit reserve. ▼

8.6 Differential Equations for Fully Continuous Benefit Reserves

In Section 8.2 a general fully discrete insurance and a general fully continuous model is developed. Section 8.3 contains the recursion relations for the fully discrete model. The parallel results for the fully continuous model are developed in this section.

The expression for the benefit reserve at t, ${}_t\bar{V}$, is given in (8.2.9) and is restated here:

$${}_t\bar{V} = \int_0^\infty b_{t+u}\, v^u\, {}_up_{x+t}\, \mu_x(t+u)\, du - \int_0^\infty \pi_{t+u}\, v^u\, {}_up_{x+t}\, du.$$

To simplify the calculation of the derivative with respect to t of ${}_t\bar{V}$, we combine the two integrals, replace the variable of integration by the substitution $s = t + u$, and then multiply inside and divide outside by the factor $v^t\, {}_tp_x$ to obtain

$${}_t\bar{V} = \frac{\int_t^\infty [b_s\, \mu_x(s) - \pi_s] v^s\, {}_sp_x\, ds}{v^t\, {}_tp_x}. \tag{8.6.1}$$

Now

$$\frac{d_t\bar{V}}{dt} = (-1)[b_t\, \mu_x(t) - \pi_t] + \frac{\mu_x(t) + \delta}{v^t\, {}_tp_x} \int_t^\infty [b_s\, \mu_x(s) - \pi_s] v^s\, {}_sp_x ds,$$

$$\frac{d_t\bar{V}}{dt} = \pi_t + [\delta + \mu_x(t)]{}_t\bar{V} - b_t\, \mu_x(t). \tag{8.6.2}$$

Here the rate of change of the benefit reserve is made up of three components: the benefit premium rate, the rate of increase of the benefit reserve under interest and survivorship, and the rate of benefit outgo. A rearrangement of formula (8.6.2) provides a formula corresponding to (8.3.12):

$$\pi_t + \delta\ {}_t\bar{V} + {}_t\bar{V}\ \mu_x(t) = b_t\ \mu_x(t) + \frac{d_t\bar{V}}{dt}. \tag{8.6.3}$$

This balances the sum of income rates to the sum of the rate of benefit outgo and the rate of change in the benefit reserve.

If the benefit reserve is treated as a savings fund available to offset the death benefit, we have

$$\pi_t + \delta\ {}_t\bar{V} = (b_t - {}_t\bar{V})\ \mu_x(t) + \frac{d_t\bar{V}}{dt}. \tag{8.6.4}$$

Here the income rates are in respect to benefit premiums and to interest on the benefit reserve, and these balance with the outgo rate, $(b_t - {}_t\bar{V})\ \mu_x(t)$, based on the net amount at risk and the rate of change in the benefit reserve. Formula (8.6.4) corresponds to (8.3.15). Again, the left side represents the resources available, benefit premiums, and investment income, and the right side represents their allocation to benefits and benefit resources.

Example 8.6.1

Use (8.6.2) to develop a retrospective formula for the benefit reserve for the general fully continuous insurance.

Solution:

We start by moving all of the benefit reserve terms of (8.6.2) to the left-hand side and then multiplying both sides by the integrating factor $\exp\{-\int_0^t[\delta + \mu_x(s)]\ ds\}$. Thus,

$$v^t\ {}_tp_x \left\{\frac{d_t\bar{V}}{dt} - [\delta + \mu_x(t)]\ {}_t\bar{V}\right\} = [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x,$$

or

$$\frac{d}{dt}\left(v^t\ {}_tp_x\ {}_t\bar{V}\right) = [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x.$$

Integration of both sides of this last equation over the interval $(0, r)$ yields

$$v^r\ {}_rp_x\ {}_r\bar{V} - {}_0\bar{V} = \int_0^r [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x\ dt.$$

For equivalence principle benefit premium rates, ${}_0V = 0$, so

$${}_r\bar{V} = \frac{\int_0^r [\pi_t - b_t\ \mu_x(t)]\ v^t\ {}_tp_x\ dt}{v^r\ {}_rp_x}. \tag{8.6.5}$$

▼

8.7 Notes and References

Recursive formulas and differential equations for the loss variables as functions of duration and the expectations and variances of these loss variables provide basic insight into long-term insurance and annuity processes. In particular, one of these recursive formulas is applied to develop Hattendorf's theorem (1868); for references, see Steffensen (1929), Hickman (1964), and Gerber (1976). This formula allocates the variance of the loss to the separate insurance years. This discussion of reserves can be easily extended to more general insurances using martingales of probability theory; see, for example, Gerber (1979). Another application of the recursion relations is to the formulation of the interim reserves at fractional durations, which is discussed for the fully discrete case in Section 8.4.

Exercises

Section 8.2

8.1. Assume that ${}_jp_x = r^j$, $b_{j+1} = 1$, $j = 0, 1, 2, 3, \ldots$, and $0 < r < 1$.
   a. If $w_0 = w_1 = w_2 = \ldots = 1$, use (8.2.6) to calculate π at the interest rate i.
   b. If $w_j = (-1)^j$, $j = 0, 1, 2, \ldots$, use (8.2.6) to calculate π at the interest rate i.

8.2. Develop a continuous analogue of (8.2.6) by applying the equivalence principle to the loss variable of (8.2.7) with $t = 0$ and $\pi_t = \pi w(t)$, where $w(t)$ is given.

8.3. If ${}_0L = T(x)\, v^{T(x)} - \pi \bar{a}_{\overline{T(x)}|}$ and the forces of mortality and interest are constant, express (a) π and (b) ${}_t\bar{V}$ in terms of μ and δ.

8.4. For the general fully discrete insurance of Section 8.2, show that for $j < h$,

$$\mathrm{Cov}(C_j, C_h) = (\pi_h - v b_{h+1}\, q_{x+h})\, {}_hp_x(\pi_j\, {}_jq_x + v b_{j+1}\, {}_jp_x\, q_{x+j}).$$

Section 8.3

8.5. Consider the life insurance policy described in Example 8.2.1. Display, for $0 < j < h$:
   a. The covariance of C_0 and C_h.
   b. Repeat part (a) for C_j and C_h.
   c. Give a rule for determining h such that the covariance of C_j and C_h is negative.

8.6. Consider the deferred annuity described in Example 8.3.1. Find $\mathrm{Cov}(C_j, C_h)$, $j < h \leq n$ and, for a fixed j, determine a condition on h such that $\mathrm{Cov}(C_j, C_h) < 0$. [Note that this condition on h does not depend on j.]

8.7. Show that (8.3.9), with h replaced by $h + 1$, can be rearranged as

$$ {}_{h+1}V = ({}_hV + \pi_h)\frac{1+i}{p_{x+h}} - b_{h+1}\frac{q_{x+h}}{p_{x+h}}. $$

Give an interpretation in words. (This is called the ***Fackler reserve*** accumulation formula, after the American actuary David Parks Fackler.)

8.8. For a fully discrete whole life insurance of 1 issued to (x), use recursion relations [(8.3.11) in (a) and (8.3.14 in (b)] to prove that

a. $ {}_kV_x = \sum_{h=0}^{k-1} \frac{P_x - vq_{x+h}}{{}_{k-h}E_{x+h}} $

b. $ {}_kV_x = \sum_{h=0}^{k-1} [P_x - vq_{x+h}(1 - {}_{h+1}V_x)](1+i)^{k-h}. $

Give an interpretation of the formulas in words.

8.9. If $b_{h+1} = {}_{h+1}V$, ${}_0V = 0$, and $\pi_h = \pi$, for $h = 0, 1, \ldots, k-1$, prove that ${}_kV = \pi\ddot{s}_{\overline{k}|}$. [Hint: Use (8.3.14).]

8.10. Show that if π is the (8.3.14) level annual benefit premium for an n-year term insurance with $b_h = \ddot{a}_{\overline{n-h}|}$, $h = 1, 2, \ldots, n$, ${}_0V = {}_nV = 0$, then

a. $ \pi = \frac{\ddot{a}_{\overline{n}|} - \ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} $

b. $ {}_kV = \ddot{a}_{\overline{n-k}|} - \ddot{a}_{x+k:\overline{n-k}|} - \pi\ddot{a}_{x+k:\overline{n-k}|}. $

[Hint: This can be shown directly or by use of (8.3.10).]

Section 8.4

8.11. Starting with (8.4.3), establish the equation

$$ {}_sp_{x+h}\,{}_{h+s}V + v^{1-s}\,{}_sq_{x+h}\,b_{h+1} = (1+i)^s({}_hV + \pi_h) \qquad 0 < s < 1. $$

Explain the result by general reasoning.

8.12. Interpret the formulas

a. $$ {}_{k+(h/m)+r}V^{(m)} \cong \left(1 - \frac{h}{m} - r\right){}_kV^{(m)} + \left(\frac{h}{m} + r\right){}_{k+1}V^{(m)} + \left(\frac{1}{m} - r\right)P^{(m)} $$

b. $$ {}_{k+(h/m)+r}V^{(m)} \cong \left(1 - \frac{h}{m} - r\right){}_kV^{(m)} + \left(\frac{h}{m} + r\right){}_{k+1}V^{(m)} + \left(\frac{1}{m} - r\right)P^{(m)}, $$

where $0 < r < 1/m$.

8.13. For each of the following benefit reserves, develop formulas similar to one or more of (8.4.8), (8.4.14), and (8.4.18).

a. ${}_{20}{}^{1}/_{2}V(\bar{A}_{x:\overline{40|}})$
b. ${}_{20}{}^{1}/_{2}V(\bar{A}_{x:\overline{40|}})$
c. ${}_{20}{}^{1}/_{2}V^{(2)}(\bar{A}_{x:\overline{40|}})$
d. ${}_{20}{}^{2}/_{3}V^{(2)}(\bar{A}_{x:\overline{40|}})$
e. ${}_{20}{}^{1}/_{2}V^{[2]}(\bar{A}_{x:\overline{40|}})$
f. ${}_{20}{}^{2}/_{3}V^{[2]}(\bar{A}_{x:\overline{40|}})$

8.14. On the basis of the Illustrative Life Table and interest of 6%, approximate ${}_{10}{}^{1}/_{6}V^{[4]}(\bar{A}_{25})$.

Section 8.5

8.15. For a fully discrete whole life insurance of amount 1 issued to (x) with premiums payable for life, show that

a. $\text{Var}[L] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)}\, {}_hp_x\, p_{x+h}\, q_{x+h}$

b. $\text{Var}[{}_kL | K(x) \geq k] = \sum_{h=0}^{\infty} \left(\frac{\ddot{a}_{x+k+h+1}}{\ddot{a}_x}\right)^2 v^{2(h+1)}\, {}_hp_{x+k}\, p_{x+k+h}\, q_{x+k+h}$.

8.16. For a life annuity-due of 1 per annum payable while (x) survives, consider the whole life loss

$$L = \ddot{a}_{\overline{K+1|}} - \ddot{a}_x \qquad K = 0, 1, 2, \ldots$$

and the loss Λ_h, valued at time h, that is allocated to annuity year h, namely,

$$\Lambda_h = \begin{cases} 0 & K \leq h-1 \\ -(\ddot{a}_{x+h} - 1) = -vp_{x+h}\, \ddot{a}_{x+h+1} & K = h \\ v\ddot{a}_{x+h+1} - (\ddot{a}_{x+h} - 1) = vq_{x+h}\, \ddot{a}_{x+h+1} & K \geq h+1. \end{cases}$$

a. Interpret the formulas for Λ_h.

b. Show that

$$\text{(i)}\ L = \sum_{h=0}^{\infty} v^h\, \Lambda_h$$

$$\text{(ii)}\ \text{E}[\Lambda_h] = 0$$

$$\text{(iii)}\ \text{Var}(\Lambda_h) = v^2\, (\ddot{a}_{x+h+1})^2\, {}_hp_x\, p_{x+h}\, q_{x+h}.$$

8.17. a. For the insurance of Example 8.3.2, establish that

$$\text{Var}(L) = \sum_{h=0}^{n-1} v^{2(h+1)}\, {}_hp_x\, p_{x+h}\, q_{x+h}.$$

b. If $\delta = 0.05$, $n = 20$, and $\mu_x(t) = 0.01$, $t \geq 0$, calculate $\text{Var}(L)$ for the insurance in (a).

8.18. A 20-payment whole life policy with unit face amount was issued on a fully discrete basis to a person age 25. On the basis of your Illustrative Life Table and interest of 6%, calculate

a. ${}_{20}P_{25}$
b. ${}^{20}_{19}V_{25}$
c. ${}^{20}_{20}V_{25}$

d. $\text{Var}[{}_{20}L|K(25) \geq 20]$ e. $\text{Var}[{}_{18}L|K(25) \geq 18]$, using Theorem 8.2.

Section 8.6

8.19. Interpret the differential equations

a. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + [\delta + \mu_x(t)]\,{}_t\bar{V} - b_t\,\mu_x(t)$

b. $\dfrac{d}{dt}\,{}_t\bar{V} = \pi_t + \delta\,{}_t\bar{V} - (b_t - {}_t\bar{V})\,\mu_x(t)$.

8.20. If $b_t = {}_t\bar{V}$, ${}_0\bar{V} = 0$, and $\pi_t = \pi$, $t \geq 0$, show that ${}_t\bar{V} = \pi\bar{s}_{\overline{t}|}$.

8.21. Evaluate $(d/dt)\,\{[1 - {}_t\bar{V}(\bar{A}_x)]\,{}_tp_x\}$.

8.22. Use (8.6.2) to write expressions for

a. $\dfrac{d}{dt}\,({}_tp_x\,{}_t\bar{V})$ b. $\dfrac{d}{dt}\,(v^t\,{}_t\bar{V})$ c. $\dfrac{d}{dt}\,(v^t\,{}_tp_x\,{}_t\bar{V})$

and interpret the results.

Miscellaneous

8.23. Show that the formula equivalent to (8.4.6) under the hyperbolic assumption for mortality within the year of age is

$${}_{k+s}V = v^{1-s}[(1 - s)({}_kV + \pi_k)(1 + i) + s\,{}_{k+1}V].$$

8.24. Prove that

$$\int_0^\infty [v^t - \bar{P}(\bar{A}_x)\bar{a}_{\overline{t}|}]^2\,{}_tp_x\,\mu_x(t)\,dt = \int_0^\infty [1 - {}_t\bar{V}(\bar{A}_x)]^2\,v^{2t}\,{}_tp_x\,\mu_x(t)\,dt$$

and interpret the result.

8.25. For a different form of the Hattendorf theorem, consider the following:

$${}_{k,m}L = \begin{cases} b_{K+1}\,v^{(K-k)+1} - {}_kV - \displaystyle\sum_{h=0}^{(K-k)} \pi_{k+h}\,v^h & K(x) = k, k+1, \ldots, k+m-1 \\ {}_{k+m}V\,v^m - {}_kV - \displaystyle\sum_{h=0}^{m-1} \pi_{k+h}\,v^h & K(x) = k+m, k+m+1, \ldots, \end{cases}$$

and, for $h = 0, 1, \ldots, m-1$,

$$\Lambda_{k+h} = \begin{cases} 0 & K(x) = k, k+1, \ldots, k+h-1 \\ vb_{k+h+1} - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h \\ v\,{}_{k+h+1}V - ({}_{k+h}V + \pi_{k+h}) & K(x) = k+h+1, k+h+2, \ldots. \end{cases}$$

Show that

a. ${}_{k,m}L = \displaystyle\sum_{h=0}^{m-1} v^h\,\Lambda_{k+h}$

b. $\text{Var}[{}_{k,m}L|K(x) \geq k] = \displaystyle\sum_{h=0}^{m-1} v^{2h}\,\text{Var}[\Lambda_{k+h}|K(x) \geq k]$.

8.26. Repeat Example 8.5.1 in terms of an insured from Example 7.4.4 who has survived to the end of the second policy year.

8.27. Repeat Example 8.5.2 in terms of a portfolio of 1,500 policies of the type described in Example 7.4.4 and discussed in Exercise 8.26.

8.28. In Exercise 8.27 there is no uncertainty about the amount or time of payment for the insureds who have survived to the end of the fourth policy year. Redo Exercise 8.27 for just those insureds at durations 2 and 3.

8.29. Write a formula, in terms of benefit premium and terminal benefit reserve symbols, for the benefit reserve at the middle of the eleventh policy year for a 10,000 whole life insurance with apportionable premiums payable annually issued to (30).

8.30. A 3-year endowment policy for a face amount of 3 has the death benefit payable at the end of the year of death and a benefit premium of 0.94 payable annually. Using an interest rate of 20%, the following benefit reserves are generated:

End of Year	Benefit Reserve
1	0.66
2	1.56
3	3.00

Calculate

a. q_x
b. q_{x+1}
c. The variance of the loss at policy issue, ${}_0L$
d. The variance of the loss at the end of the first year, ${}_1L$.

8.31. a. Use (8.3.10) to transform (8.5.1) to

$$\Lambda_h = \begin{cases} 0 & K(x) \le h-1 \\ (b_{h+1} - {}_{h+1}V)\, v - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) = h \\ 0 - (b_{h+1} - {}_{h+1}V)\, v\, q_{x+h} & K(x) \ge h+1. \end{cases}$$

In this interpretation Λ_h is the loss on the 1-year term insurance for the amount at risk in the year $(h, h+1)$.

b. Use the display in (a) to verify $E[\Lambda_h] = 0$.
c. Use parts (a) and (b) to obtain $\text{Var}(\Lambda_h)$.

Computing Exercises

8.32. Consider a variation of the insurance of Example 8.2.2 which provides a unit benefit whole life insurance to (20) with geometrically increasing benefit premiums payable to age 65. On the basis of your Illustrative Life Table and $i = 0.06$, determine the maximum value of r such that the benefit reserve is non-negative at all durations.

8.33. Use the backward recursion formula (8.3.9) to calculate the benefit reserves of
   a. Example 7.4.3 [Hint: ${}_5V = 0.0$.]
   b. Example 7.4.4. [Hint ${}_5V = 1.0$.]

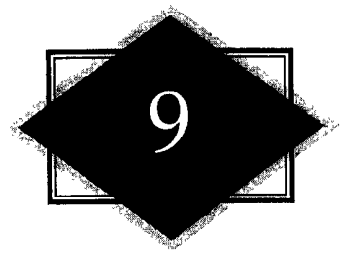
8.34. A decreasing term insurance to age 65 with immediate payment of death claims is issued to (30) with the following benefits:

For Death between Ages	Benefit
30–50	100,000
50–55	90,000
55–60	80,000
60–65	60,000

On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $i = 0.06$, determine
   a. The annual apportionable benefit premium, payable semiannually and
   b. The reserve at the end of 30 years, if benefit premiums are as in (a).

8.35. A single premium insurance contract issued to (35) provides 100,000 in case the insured survives to age 65, and it returns (at the end of the year of death) the single benefit premium without interest if the insured dies before age 65. If the single benefit premium is denoted by S, write expressions, in terms of actuarial functions, for
   a. S
   b. The prospective formula for the benefit reserve at the end of k years
   c. The retrospective formula for the benefit reserve at the end of k years
   d. On the basis of your Illustrative Life Table and $d = 0.05$, calculate S and the benefit reserve ${}_{20}V$.

8.36. In terms of $P = {}_{20}P^{(12)}(\bar{A}_{30:\overline{35}|})$ and actuarial functions, write prospective and retrospective formulas for the following:
   a. ${}^{20}_{10}V^{(12)}(\bar{A}_{30:\overline{35}|})$
   b. ${}^{20}_{25}V^{(12)}(\bar{A}_{30:\overline{35}|})$
   c. On the basis of your Illustrative Life Table with uniform distribution of deaths within each year of age and $\delta = 0.05$, calculate P and the benefit reserves of parts (a) and (b).



MULTIPLE LIFE FUNCTIONS

9.1 Introduction

In Chapters 3 through 8 we developed a theory for the analysis of financial benefits contingent on the time of death of a single life. We can extend this theory to benefits involving several lives. An application of this extension commonly found in pension plans is the joint-and-survivor annuity option. Other applications of multiple life actuarial calculations are common. In estate and gift taxation, for example, the investment income from a trust can be paid to a group of heirs as long as at least one of the group survives. Upon the last death, the principal from the trust is to be donated to a qualified charitable institution. The amount of the charitable deduction allowed for estate tax purposes is determined by an actuarial calculation. There are family policies in which benefits differ due to the order of the deaths of the insured and the spouse, and there are insurance policies with benefits payable on the first or last death providing cash in accordance with an estate plan.

In this chapter we discuss models involving two lives. Actuarial present values for basic benefits are derived by applying the concepts and techniques developed in Chapters 3 through 5. Models built on the assumption that the two future lifetime random variables are independent constitute most of the chapter. Section 9.6 introduces special models in which the two future lifetime random variables are dependent. Annual benefit premiums, reserves, and models involving three or more lives are covered in Chapter 18.

A useful abstraction in the theory of life contingencies, particularly as it is applied to several lives, is that of ***status*** for which there are definitions of survival and failure. Two elements are necessary for a status to be defined. The general term ***entities*** is used in the definition because of the broad range of application of the concept:

- There must be a finite set of entities, and for each member it must be possible to define a future lifetime random variable.
- There must be a rule by which the survival of the status can be determined at any future time.

To compute probabilities or actuarial present values associated with the survival of a status, the joint distribution of the future lifetime random variables must be available. Some of these random variables may have a marginal distribution such that all the probability is at one point.

Several illustrations of the status concept may be helpful. A single life age x defines a status that survives while (x) lives. Thus, the random variable $T(x)$, used in Chapter 3 to denote the future lifetime of (x), can be interpreted as the period of survival of the status and also as the time-until-failure of the status. A term certain, $\overline{n|}$, defines a status surviving for exactly n years and then failing. More complex statuses can be defined in terms of several lives in various ways. Survival can mean that all members survive or, alternatively, that at least one member survives. Still more complicated statuses can be in regard to two men and two women with the status considered to survive only as long as at least one man and at least one woman survive.

After a status and its survival have been defined, we can apply the definition to develop models for annuities and insurances. An annuity is payable as long as the status survives, whereas an insurance is payable upon the failure of the status. Insurances also can be restricted so they are payable only if the individuals die in a specific order.

9.2 Joint Distributions of Future Lifetimes

The time-until-failure of a status is a function of the future lifetimes of the lives involved. In theory these future lifetimes will be dependent random variables. We will explore the consequences of that dependence. For convenience, or because of the lack of data on dependent lives, in practice, an assumption of independence among the future lifetimes has traditionally been made. With the independence assumption numerical values from the marginal distributions (life tables) for single lives can be used.

Example 9.2.1

While the distribution in this example is not realistic, it is offered as a vehicle to explore a joint distribution for two dependent future lifetimes. For two lives (x) and (y), the joint p.d.f. of their future lifetimes, $T(x)$ and $T(y)$, is

$$f_{T(x)T(y)}(s, t) = \begin{cases} 0.0006(t - s)^2 & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the following:

a. The joint d.f. of $T(x)$ and $T(y)$

b. The p.d.f., d.f., ${}_sp_x$, and $\mu(x + s)$ for the marginal distribution of $T(x)$. Note the symmetry of the distribution in s and t, which implies that $T(x)$ and $T(y)$ are identically distributed.

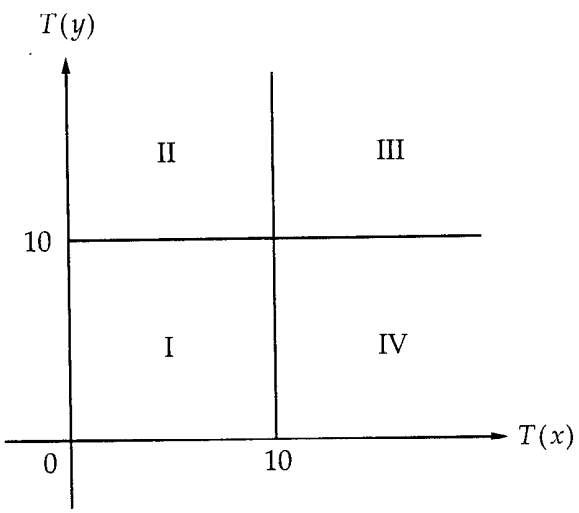
c. The correlation coefficient of $T(x)$ and $T(y)$.

Solution:

a. Before calculating, we look at the sample space of $T(x)$ and $T(y)$ in Figure 9.2.1 and observe the region where the joint p.d.f. is positive. At points outside the first quadrant, the d.f. will be 0. In the first quadrant we start by calculating the d.f. at a point in Region I where both s and t are between 0 and 10:

$$\begin{aligned} F_{T(x)T(y)}(s, t) &= \Pr[T(x) \le s \text{ and } T(y) \le t] \\ &= \int_{-\infty}^{s} \int_{-\infty}^{t} f_{T(x)T(y)}(u, v)\, dv\, du \\ &= \int_{0}^{s} \int_{0}^{t} 0.0006(v - u)^2\, dv\, du \\ &= 0.00005[s^4 + t^4 - (t - s)^4] \\ &\quad 0 < s \le 10,\ 0 < t \le 10. \end{aligned}$$

Sample Space of $T(x)$ and $T(y)$



Sample Space of $T(x)$ and $T(y)$

Since the joint p.d.f. is 0 in regions II, III, and IV, we have

$$\left.\begin{aligned} F_{T(x)T(y)}(s, t) &= F_{T(x)T(y)}(s, 10) = F_{T(x)}(s) \\ &= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] \end{aligned}\right\} \text{ in Region II}$$

$$\left.\begin{aligned} &= F_{T(x)T(y)}(10, t) = F_{T(y)}(t) \\ &= \frac{1}{2} + 0.00005[t^4 - (10 - t)^4] \end{aligned}\right\} \text{ in Region IV}$$

$$= 1 \qquad \text{in Region III.}$$

b. Using the d.f. obtained in part (a), we have

$$\begin{aligned} F_{T(x)T(y)}(s, 10) &= F_{T(x)}(s) \\ &= 0 && s \le 0 \\ &= \frac{1}{2} + 0.00005[s^4 - (10 - s)^4] && 0 < s \le 10 \\ &= 1 && s > 10 \end{aligned}$$

and

$$f_{T(x)}(s) = F'_{T(x)}(s) = \begin{cases} 0.0002[s^3 + (10 - s)^3] & 0 < s \le 10 \\ 0 & \text{elsewhere.} \end{cases}$$

The survival probability and force of mortality are given by

$$\begin{aligned} {}_sp_x &= 1 - F_{T(x)}(s) \\ &= \frac{1}{2} + 0.00005[(10 - s)^4 - s^4] && 0 < s \le 10 \\ &= 0 && s > 10, \end{aligned}$$

and

$$\begin{aligned} \mu(x + t) &= \frac{f_{T(x)}(t)}{1 - F_{T(x)}(t)} \\ &= \frac{0.0002[s^3 + (10 - s)^3]}{1/2 + 0.00005[(10 - s)^4 - s^4]} && 0 < s \le 10. \end{aligned}$$

c.
$$\mathrm{E}[T(x)] = \int_0^{10} s(0.0002)[s^3 + (10 - s)^3]\, ds = 5 = \mathrm{E}[T(y)],$$

$$\mathrm{E}[T(x)^2] = \int_0^{10} s^2(0.0002)[s^3 + (10 - s)^3]\, ds = \frac{110}{3} = \mathrm{E}[T(y)^2],$$

$$\mathrm{Var}[T(x)] = \frac{35}{3} = \mathrm{Var}[T(y)],$$

$$\mathrm{E}[T(x)T(y)] = \int_0^{10} \int_0^{10} st(0.0006)(t - s)^2\, ds\, dt = \frac{50}{3},$$

$$\mathrm{Cov}[T(x), T(y)] = \mathrm{E}[T(x)T(y)] - \mathrm{E}[T(x)]\mathrm{E}[T(y)] = -\frac{25}{3},$$

$$\rho_{T(x)T(y)} = \frac{\mathrm{Cov}[T(x), T(y)]}{\sigma_{T(x)}\sigma_{T(y)}} = \frac{-25/3}{35/3} = -\frac{5}{7}.$$ ▼

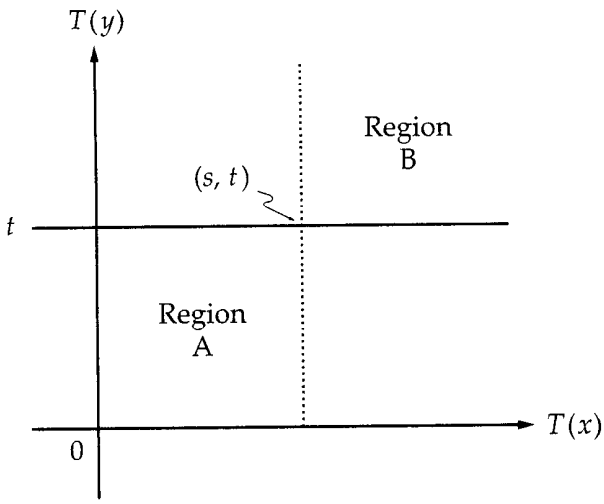
For the joint life distribution, we define the ***joint survival function*** as

$$s_{T(x)T(y)}(s, t) = \Pr[T(x) > s \text{ and } T(y) > t]. \qquad (9.2.1)$$

Unlike the single life distribution, the d.f. and survival function do not necessarily add up to 1. Their relationship for the joint life distribution can be illustrated by

a graph of their joint sample space as shown in Figure 9.2.2. The d.f. $F_{T(x)\,T(y)}(s, t)$ gives the probability of Region A, "southwest" of the point (s, t), and $s_{T(x)\,T(y)}(s, t)$ gives the probability of Region B, "northeast" of (s, t).

Sample Space of Future Lifetime Random Variables $T(x)$ and $T(y)$



Sample Space of Future Lifetime
Random Variables $T(x)$ and $T(y)$

Example 9.2.2

For the distribution of $T(x)$ and $T(y)$ in Example 9.2.1 determine the joint survival function.

Solution:

For $0 < s < 10$ and $0 < t < 10$,

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= \Pr[T(x) > s \cap T(y) > t] \\ &= \int_s^{\infty} \int_t^{\infty} f_{T(x)T(y)}(u, v)\, dv\, du \\ &= \int_s^{10} \int_t^{10} 0.0006(v - u)^2\, dv\, du \\ &= 0.00005[(10 - t)^4 + (10 - s)^4 - (t - s)^4]. \end{aligned}$$

For other points in the first quadrant, $s_{T(x)\,T(y)}(s, t)$ will be 0 and for all points in the third quadrant it will be 1. In the second quadrant, where $s < 0$ and $t > 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(y)}(t) = {}_tp_y.$$

In the fourth quadrant, where $s > 0$ and $t < 0$,

$$s_{T(x)\,T(y)}(s, t) = s_{T(x)}(s) = {}_tp_x.$$

▼

In Example 9.2.1 we were given the joint distribution of two dependent future lifetimes and then determined their marginal distributions and their correlation coefficient, which indicated their degree of dependence. In applications, the dependence of the time-until-death random variables may be difficult to quantify. Consequently, the future lifetimes are usually assumed to be independent, and then their joint distribution is obtained from their marginal single life distributions that we discussed in Chapter 3. This is illustrated in the next example.

Example 9.2.3

The future lifetimes $T(x)$ and $T(y)$ are independent, and each has the distribution defined by the p.d.f.

$$f(t) = \begin{cases} 0.02\ (10 - t) & 0 < t < 10 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Determine the d.f., survival function, and force of mortality of this distribution.
b. Determine the joint p.d.f., d.f., and survival function for $T(x)$ and $T(y)$.

Solution:

a. $$F_{T(x)}(t) = \int_{-\infty}^{t} f_{T(x)}(s)\, ds$$

$$= \begin{cases} 0 & t \le 0 \\ 1 - 0.01(10 - t)^2 = 0.2t - 0.01t^2 & 0 < t \le 10 \\ 1 & t > 10, \end{cases}$$

$$s_{T(x)}(t) = 1 - F_{T(x)}(t) = \begin{cases} 1 & t < 0 \\ 0.01(10 - t)^2 & 0 \le t < 10 \\ 0 & t \ge 10, \end{cases}$$

$$\mu(x + t) = \frac{f_{T(x)}(t)}{s_{T(x)}(t)} = \frac{2}{10 - t} \qquad 0 < t < 10.$$

b. $$f_{T(x)T(y)}(s, t) = f_{T(x)}(s) f_{T(y)}(t)$$

$$= \begin{cases} (0.02)^2(10 - s)(10 - t) & 0 < s < 10,\ 0 < t < 10 \\ 0 & \text{elsewhere,} \end{cases}$$

$$F_{T(x)T(y)}(s, t) = F_{T(x)}(s) F_{T(y)}(t)$$

$$= (0.2)^2(t - 0.05t^2)(s - 0.05s^2) \qquad 0 < s \le 10,\ 0 < t \le 10$$

$$= F_{T(x)}(s) = (0.2)(s - 0.05s^2) \qquad 0 < s \le 10,\ t > 10$$

$$= F_{T(y)}(t) = (0.2)(t - 0.05t^2) \qquad s > 10,\ 0 < t \le 10,$$

$$s_{T(x)T(y)}(s, t) = s_{T(x)}(s) s_{T(y)}(t)$$

$$= (0.01)^2(10 - s)^2(10 - t)^2 \qquad 0 \le s < 10,\ 0 \le t < 10$$

$$= s_{T(x)}(s) = (0.01)(10 - s)^2 \qquad 0 \le s < 10,\ t < 0$$

$$= s_{T(y)}(t) = (0.01)(10 - t)^2 \qquad s < 0,\ 0 \le t < 10$$

$$= 0 \qquad s \ge 10,\ t \ge 10. \quad \blacktriangledown$$

9.3 The Joint-Life Status

A status that survives as long as all members of a set of lives survive and fails upon the first death is called a ***joint-life status.*** It is denoted by $(x_1, x_2, \ldots, x_m)$, where x_i represents the age of member i of the set and m represents the number of members. Notation introduced in Chapters 3 through 5 is used here with the subscript listing several ages rather than a single age. For example, A_{xy} and ${}_tp_{xy}$ have the same meaning for the joint-life status (xy) as A_x and ${}_tp_x$ have for the single life (x).

A joint-life status is an example of what we call a ***survival status,*** that is, a status for which there is a future lifetime random variable, and, therefore, a survival function can be defined. For the future lifetime of a survival status, the concepts and relationships established in Sections 3.2.2 through 3.5 (excluding the life table example in Section 3.3.2) apply to the distribution of the survival status. These concepts will be used here without new proofs.

We now consider the distribution of the time-until-failure of a joint-life status. For m lives, $T(x_1, x_2, \ldots, x_m) = \min[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time of death of individual i. For the special case of two lives, (x) and (y), we have $T(xy) = \min[T(x), T(y)]$. When clear by context, we denote the future lifetime of the joint-life status by simply T. The student can interpret the time-until-failure of the joint-life status as the ***smallest order statistic*** of the m lifetimes in the set. In previous studies of order statistics, the random variables in the sample have usually been independent and identically distributed. Here the random variables are typically independent by assumption but are rarely identically distributed.

We begin by expressing the distribution function of T, for $t > 0$, in terms of the joint distribution of $T(x)$ and $T(y)$ for the general (dependent) case:

$$\begin{aligned} F_T(t) &= {}_tq_{xy} = \Pr(T \le t) \\ &= \Pr\{\min[T(x), T(y)] \le t\} \\ &= 1 - \Pr\{\min[T(x), T(y)] > t\} \\ &= 1 - \Pr\{T(x) > t \text{ and } T(y) > t\} \\ &= 1 - s_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.1}$$

Another equation can be obtained from the second line by recognizing that the event $\{\min[T(x), T(y)] \le t\}$ is the union of $\{T(x) \le t\}$ and $\{T(y) \le t\}$. Then,

$$F_T(t) = \Pr\{\min[T(x), T(y)] \le t\},$$

and using a basic result in probability, we have

$$\begin{aligned} F_T(t) &= \Pr[T(x) \le t] + \Pr[T(y) \le t] - \Pr[T(x) \le t \cap T(y) \le t] \\ &= {}_tq_x + {}_tq_y - F_{T(x)\,T(y)}(t, t). \end{aligned} \tag{9.3.2}$$

The mixture of IAN and standard probability/statistics notation in (9.3.2) demonstrates that although the IAN system may accommodate survival statuses, it does not provide for the joint distribution of several statuses except in the independent case which can be expressed in single survival status symbols.

When $T(x)$ and $T(y)$ are independent, the two expressions for the d.f. of T can be written in terms of single life functions as:

$$F_T(t) = \Pr\{\min[T(x), T(y)] \le t\}$$

$$= 1 - s_{T(x)\ T(y)}(t, t) = 1 - {}_tp_x\ {}_tp_y, \tag{9.3.3}$$

and

$$F_T(t) = {}_tq_x + {}_tq_y - F_{T(x)\ T(y)}(t, t) = {}_tq_x + {}_tq_y - {}_tq_x\ {}_tq_y. \tag{9.3.4}$$

The survival function for the joint-life status, ${}_tp_{xy}$, is obtained by subtracting the d.f. from 1.

For the general case, ${}_tp_{xy} = s_{T(x)\ T(y)}(t, t)$ using (9.3.1). In the independent case, we have by (9.3.3)

$${}_tp_{xy} = {}_tp_x\ {}_tp_y. \tag{9.3.5}$$

Expression (9.3.5) is the convenient starting point for the independent case since the joint-life status survives to t if, and only if, both (x) and (y) survive to t.

Example 9.3.1

Determine the d.f., survival function, and complete expectation for the joint-life status, $T(xy)$, for the lives of Example 9.2.1.

Solution:

For $t \le 0$ and $t > 10$, the value of $F_{T(xy)}(t)$ would be 0 and 1, respectively. For $0 < t \le 10$, we have by the results of Example 9.2.1(a) and (b) and (9.3.2)

$$F_{T(xy)}(t) = 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - 0.0001\ t^4$$

$$= 1 - 0.0001\ (10 - t)^4$$

for $0 < t \le 10$.

Now,

$${}_tp_{xy} = 1 - F_{T(xy)}(t) = 0.0001(10 - t)^4 \qquad 0 \le t < 10.$$

From (3.5.2),

$$\mathring{e}_{xy} = \mathrm{E}[T(xy)] = \int_0^\infty {}_tp_{xy}\, dt = \int_0^{10} 0.0001(10 - t)^4\, dt = 2. \qquad \blacktriangledown$$

Example 9.3.2

Determine the d.f. and survival function and the complete expectation for the joint-life status, $T(xy)$, for the distribution of Example 9.2.3.

Solution:

For independent lives we use (9.3.5) and the results of Example 9.2.3 to obtain

$$ {}_tp_{xy} = [0.01(10 - t)^2]^2 = 0.0001(10 - t)^4 \qquad \text{for } 0 \le t < 10. $$

Then

$$ F_{T(xy)}(t) = 1 - (0.0001)(10 - t)^4 \qquad \text{for } 0 < t \le 10 $$

and

$$ \mathring{e}_{xy} = \int_0^{10} 0.0001(10 - t)^4 \, dt = 2. $$

▼

An insight can be gained from the two previous examples. Although the joint distributions of the two underlying future lifetime random variables of the two examples are not the same, the distributions of their joint-life statuses are the same. This is a important point in practice when only the first-to-die can be observed. In such case the underlying joint distribution is not uniquely determined. In statistics this is called ***nonidentifiability*** because of the difficulty in distinguishing among two or more models for the same observed data.

The p.d.f. for T can be obtained by differentiating its d.f. as displayed in either (9.3.1) or (9.3.2). For (9.3.1) we will need the derivative, with respect to t, of

$$ s_{T(x)\,T(y)}(t, t) = \int_t^\infty \int_t^\infty f_{T(x)T(y)}(u, v) \, du \, dv. $$

Using the formula from calculus in Appendix 5, we have

$$ \frac{d}{dt} s_{T(x)\,T(y)}(t, t) = -\left[\int_t^\infty f_{T(x)T(y)}(t, v) \, dv + \int_t^\infty f_{T(x)T(y)}(u, t) \, du \right]. $$

Hence,

$$ f_{T(xy)}(t) = \int_t^\infty f_{T(x)T(y)}(t, v) \, dv + \int_t^\infty f_{T(x)T(y)}(u, t) \, du. \tag{9.3.6} $$

Using (9.3.2), the reader can show that the p.d.f. of T can also be written as

$$ f_{T(xy)}(t) = f_{T(x)}(t) + f_{T(y)}(t) - \left[\int_0^t f_{T(x)T(y)}(t, v) \, dv + \int_0^t f_{T(x)T(y)}(u, t) \, du \right], $$

or with actuarial notation as

$$ f_{T(xy)}(t) = {}_tp_x \, \mu(x + t) + {}_tp_y \, \mu(y + t) - \left[\int_0^t f_{T(x)T(y)}(t, v) \, dv + \int_0^t f_{T(x)T(y)}(u, t) \, du \right]. \tag{9.3.7} $$

When $T(x)$ and $T(y)$ are independent, $f_{T(x)T(y)}(u, v) = {}_up_x \, \mu(x + u) \, {}_vp_y \, \mu(y + v)$, and (9.3.6) reduces directly to

$$ = {}_tp_y \, {}_tp_x \, [\mu(x + t) + \mu(y + t)]. \tag{9.3.8} $$

Example 9.3.3

By use of (9.3.6) determine the p.d.f. of $T(xy)$ for Example 9.2.1. Verify your result by examination of the d.f. in Example 9.3.1.

Solution:

Using (9.3.6) we obtain

$$f_T(t) = \begin{cases} \displaystyle\int_t^{10} 0.0006\,(t-v)^2\,dv + \int_t^{10} 0.0006\,(u-t)^2\,du \\ \qquad = 0.0004(10-t)^3 & \text{for } 0 < t < 10, \\ 0 & \text{elsewhere.} \end{cases}$$

This is the derivative of the d.f. in Example 9.3.1. ▼

We saw that $T(xy)$ has the same distribution in Examples 9.3.1 and 9.3.2. If we use (9.3.8) to obtain the p.d.f. of $T(xy)$ for Example 9.2.3, we will see this again. This is left as Exercise 9.8.

As explained in Chapter 3, the distribution of $T = T(xy)$ can also be specified by the force of "mortality," or more generally, the force of "failure." First we consider a notation for the force of failure of the status at time t. The traditional notation for this force is $\mu_{x+t:y+t}$ (in analogy with μ_{x+t}), but, in preparation for discussing other statuses where duration must be recognized, and in accordance with the notational convention adopted in Chapter 3, we use the notation $\mu_{xy}(t)$. The notation $\mu_{xy}(t)$ does not necessarily mean that (x) and (y) or the survival status (xy) were subject to a selection process, but the status did come into existence at these ages.

By analogy with the first formula of (3.2.12) and with $f_{T(x)}(x)$ and $F_{T(x)}(x)$ replaced by $f_{T(xy)}(t)$ and $F_{T(xy)}(t)$, we have

$$\mu_{xy}(t) = \frac{f_{T(xy)}(t)}{1 - F_{T(xy)}(t)}. \tag{9.3.9}$$

For dependent $T(x)$, $T(y)$, this expression does not simplify beyond the general form. However, using (9.3.3) and (9.3.8) for the independent case, we have $\mu_{xy}(t) = \mu(x+t) + \mu(y+t)$.

In words, if the future lifetimes are independent, the force of failure for their joint-life status is the sum of the forces of mortality for the individuals. As in Chapter 3 with the single life case, we can characterize the distribution of $T(xy)$ by the p.d.f., the d.f., the survival function, or the force of failure.

Example 9.3.4

Determine the force of failure for the joint-life statuses of (x) and (y) in Examples 9.2.1 and 9.2.3.

Solution:

Since the joint distributions of the two examples produce the same distribution for $T(xy)$, we will use the independent case. From the results of part (a) of Example 9.2.3 and (9.3.9),

$$\mu_{xy}(t) = \frac{4}{10 - t} \qquad 0 < t < 10.$$

▼

We now turn to the curtate future lifetime of the joint-life status.

The probability that the joint-life status fails during the time k to $k + 1$ is determined by

$$\begin{aligned} \Pr(k < T \leq k + 1) &= \Pr(T \leq k + 1) - \Pr(T \leq k) \\ &= {}_kp_{xy} - {}_{k+1}p_{xy} \\ &= {}_kp_{xy}\, q_{x+k:y+k}. \end{aligned} \tag{9.3.10}$$

When the future lifetimes of (x) and (y) are independent, the probability of the joint-life status $(x + k:y + k)$ failing within the next year can be written in terms of the probabilities of independent failure of the individual lives as follows:

$$\begin{aligned} q_{x+k:y+k} &= 1 - p_{x+k:y+k} \\ &= 1 - p_{x+k}\, p_{y+k} \\ &= 1 - (1 - q_{x+k})(1 - q_{y+k}) \\ &= q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k} \\ &= q_{x+k} + (1 - q_{x+k})\, q_{y+k}. \end{aligned} \tag{9.3.11}$$

From the discussion of curtate-future-lifetime of (x) in Section 3.2.3, we see that (9.3.10) also provides the p.f. of the random variable K, the number of years completed prior to failure of the joint-life status; that is, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \Pr(K = k) &= \Pr(k \leq T < k + 1) \\ &= \Pr(k < T \leq k + 1) \\ &= {}_kp_{xy}\, q_{x+k:y+k} \\ &= {}_{k|}q_{xy}. \end{aligned} \tag{9.3.12}$$

Example 9.3.5

Determine the p.f. and curtate expectation of $K(xy)$ using the common d.f. of Examples 9.3.1 and 9.3.2.

Solution:

From Example 9.3.2,

$${}_kp_{xy} = 0.0001(10 - k)^4.$$

Hence

$$\Pr[K(xy) = k] = 0.0001[(10 - k)^4 - (9 - k)^4], \quad k = 0, 1, 2, 3, \ldots, 9.$$

From (3.5.5),

$$e_{xy} = \mathrm{E}[K(xy)] = \sum_{k=0}^{\infty} {}_{k+1}p_{xy} = \sum_{k=0}^{9} 0.0001(9 - k)^4 = 1.5333.$$

▼

9.4 The Last-Survivor Status

In addition to benefits defined in terms of the time of the first death, there are those defined in terms of the time of the last death. In this section we will examine situations in which the random variable is the time of the last death.

A survival status that exists as long as at least one member of a set of lives is alive and fails upon the last death is called the ***last-survivor status***. It is denoted by $(\overline{x_1\ x_2 \cdots x_m})$, where x_i represents the age of member i and m represents the number of members of the set. We consider the distribution of the time-until-failure of the last-survivor status, the random variable $T = \max[T(x_1), T(x_2), \ldots, T(x_m)]$, where $T(x_i)$ is the time-until-death of individual i. The random variable T can be interpreted as the ***largest order statistic*** associated with $[T(x_1), T(x_2), \ldots, T(x_m)]$. Unlike the typical situation in the study of inferential statistics, the m random variables here are not necessarily independent and identically distributed.

For the case of two lives (x) and (y), $T(\overline{xy}) = \max[T(x), T(y)]$. General relationships exist among $T(xy)$, $T(\overline{xy})$, $T(x)$, and $T(y)$. For each outcome, $T(xy)$ equals either $T(x)$ or $T(y)$ and $T(\overline{xy})$ equals the other. Thus, for all joint distributions of $T(x)$ and $T(y)$, the following relationships hold:

$$T(xy) + T(\overline{xy}) = T(x) + T(y), \tag{9.4.1}$$

$$T(xy)\ T(\overline{xy}) = T(x)\ T(y), \tag{9.4.2}$$

$$a^{T(xy)} + a^{T(\overline{xy})} = a^{T(x)} + a^{T(y)} \quad \text{for } a > 0. \tag{9.4.3}$$

There are also some general relationships among the distributions of these four random variables that come from the method of inclusion-exclusion of probability; that is,

$$\Pr(A \cup B) + \Pr(A \cap B) = \Pr(A) + \Pr(B). \tag{9.4.4}$$

Defining A as $\{T(x) \le t\}$ and B as $\{T(y) \le t\}$, we have $A \cap B = \{T(\overline{xy}) \le t\}$ and $A \cup B = \{T(xy) \le t\}$, which lead to

$$F_{T(xy)}(t) + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t). \tag{9.4.5}$$

From this it follows that

$${}_tp_{xy} + {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y \tag{9.4.6}$$

and

$$f_{T(xy)}(t) + f_{T(\overline{xy})}(t) = f_{T(x)}(t) + f_{T(y)}(t). \tag{9.4.7}$$

The relationships developed above allow the distribution of the last-survivor status to be explored by use of the distribution of the joint-life status that is developed in the previous section. An illustration of this fact is provided by substituting (9.3.2) into (9.4.5) to obtain

$$F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(x)}(t) - F_{T(xy)}(t) = F_{T(x)T(y)}(t, t),$$

a relationship that also follows from $F_{T(\overline{xy})}(t) = \Pr[T(x) \le t \cap T(y) \le t]$.

Example 9.4.1

Determine the d.f., survival function of p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.1.

Solution:

From (9.4.5) and the solutions to part (b) of Example 9.2.1 and Example 9.3.1,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2\{0.5 + 0.00005[t^4 - (10 - t)^4]\} - [1 - 0.0001(10 - t)^4] \\ &= 0.0001t^4 = F_{T(x)T(y)}(t, t) \qquad 0 \le t < 10, \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - 0.0001t^4 \qquad 0 < t \le 10. \end{aligned}$$

By differentiation,

$$f_{T(\overline{xy})}(t) = 0.0004\, t^3 \qquad 0 < t < 10. \qquad \blacktriangledown$$

Example 9.4.2

Determine the d.f., survival function and p.d.f. of $T(\overline{xy})$ for the lives in Example 9.2.3.

Solution:

From (9.4.5) and the solutions to Examples 9.2.3 and 9.3.2 for $0 < t \le 10$,

$$\begin{aligned} F_{T(\overline{xy})}(t) &= 2[1 - 0.01(10 - t)^2] - [1 - 0.0001(10 - t)^4] \\ &= [1 - 0.01(10 - t)^2]^2 \\ &= t^2(0.2 - 0.01t)^2 = F_{T(x)T(y)}(t, t), \\ {}_tp_{\overline{xy}} &= 1 - F_{T(\overline{xy})}(t) = 1 - t^2(0.2 - 0.01t)^2, \\ f_{T(\overline{xy})}(t) &= 0.04t(2 - 0.1t)(1 - 0.1t) \qquad 0 < t < 10. \end{aligned}$$

$\blacktriangledown$

An observation derived by comparing Examples 9.3.1 and 9.3.2 with Examples 9.4.1 and 9.4.2 is that two different joint distributions may produce the same distribution for the joint-life status but different distributions for the last-survivor status. This possibility could have been anticipated by the general nature of (9.4.5).

For applications it is preferable to rearrange and to restate (9.4.5) and (9.4.7) in actuarial notation:

$$ {}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}, \qquad \text{(9.4.5) restated} $$

$$ {}_tp_{\overline{xy}}\,\mu_{\overline{xy}}(t) = {}_tp_x\,\mu(x+t) + {}_tp_y\,\mu(y+t) - {}_tp_{xy}\,\mu_{xy}(t). \qquad \text{(9.4.7) restated} $$

The force of failure for the last-survivor status is implicitly defined in this restatement of (9.4.7) as

$$ \mu_{\overline{xy}}(t) = \frac{f_{T(\overline{xy})}(t)}{1 - F_{T(\overline{xy})}(t)} $$

$$ = \frac{{}_tp_x\,\mu(x+t) + {}_tp_y\,\mu(y+t) - {}_tp_{xy}\,\mu_{xy}(t)}{{}_tp_{\overline{xy}}}. \qquad (9.4.8) $$

When $T(x)$ and $T(y)$ are independent $\mu_{xy}(t)$ can be replaced by $\mu_x(t) + \mu_y(t)$ in (9.4.7) and (9.4.8), and they can be rewritten as

$$ {}_tp_{\overline{xy}}\,\mu_{\overline{xy}}(t) = {}_tq_y\,{}_tp_x\,\mu(x+t) + {}_tq_x\,{}_tp_y\,\mu(y+t) \qquad (9.4.9) $$

and

$$ \mu_{\overline{xy}}(t) = \frac{{}_tq_y\,{}_tp_x\,\mu(x+t) + {}_tq_x\,{}_tp_y\,\mu(y+t)}{{}_tq_y\,{}_tp_x + {}_tq_x\,{}_tp_y + {}_tp_x\,{}_tp_y}. \qquad (9.4.10) $$

In this form the force of failure of the last-survivor status is a weighted average of forces of mortality. Forces of mortality are conditional p.d.f.'s, and the probability density that (x) and (y) will die at t is 0. As a result, the force associated with ${}_tp_x\,{}_tp_y$ in the weighted average is 0.

Example 9.4.3

Determine the force of failure for the last survivor of the two lives in
a. Example 9.2.1, and
b. Example 9.2.3.

Solution:

a. Using the results of Example 9.4.1,

$$ \mu_{\overline{xy}}(t) = \frac{0.0004t^3}{1 - 0.0001t^4} = \frac{4t^3}{10{,}000 - t^4}. $$

b. Using the results of Example 9.4.2,

$$ \mu_{\overline{xy}}(t) = \frac{0.04t(2 - 0.1t)(1 - 0.1t)}{1 - t^2(0.2 - 0.01t)^2} = \frac{4t(20 - t)(10 - t)}{10{,}000 - t^2(20 - t)^2}. $$ ▼

Discrete analogues of relationships (9.4.1)–(9.4.3) and (9.4.5)–(9.4.7) exist for the curtate-future-lifetimes. These are

$$ K(xy) + K(\overline{xy}) = K(x) + K(y), \qquad (9.4.11) $$

$$ K(xy)\,K(\overline{xy}) = K(x)\,K(y), \qquad (9.4.12) $$

$$a^{K(xy)} + a^{K(\overline{xy})} = a^{K(x)} + a^{K(y)} \qquad \text{for } a > 0, \tag{9.4.13}$$

$$F_{K(xy)}(k) + F_{K(\overline{xy})}(k) = F_{K(x)}(k) + F_{K(y)}(k). \tag{9.4.14}$$

From (9.4.14) it follows that

$$f_{K(xy)}(k) + f_{K(\overline{xy})}(k) = f_{K(x)}(k) + f_{K(y)}(k). \tag{9.4.15}$$

The distribution of $K(\overline{xy})$, the number of years completed prior to failure of the last-survivor status, that is, the number of years completed prior to the last death, can now be determined from these relationships and the results for the curtate-future-lifetime of the joint-life survivor status. From (9.4.15),

$$\Pr[K(\overline{xy}) = k] = f_{K(\overline{xy})}(k) = {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}. \tag{9.4.16}$$

For independent lives, (9.3.5) and (9.3.12) allow us to write (9.4.16) as

$$\begin{aligned}\Pr[K(\overline{xy}) = k] &= {}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_x\, {}_kp_y\, (q_{x+k} + q_{x+k} - q_{x+k}q_{y+k}) \\ &= (1 - {}_kp_y)\, {}_kp_x\, q_{x+k} + (1 - {}_kp_x)\, {}_kp_y\, q_{y+k} + {}_kp_x\, {}_kp_y\, q_{x+k}\, q_{x+k}.\end{aligned}$$

In this last form, the first two terms are the probability that only the second death occurs between times k and $k + 1$. The third term is the probability that both deaths occur during that year. This expression for $\Pr[K(\overline{xy}) = k]$ is analogous to (9.4.9) for the p.d.f. of $T(\overline{xy})$ where, since the probability that two deaths occur in the same instant is 0, there are only two terms.

9.5 More Probabilities and Expectations

In Sections 9.3 and 9.4 we express the p.d.f.'s and the d.f.'s of the future lifetimes of the joint-life status and the last-survivor status in terms of the functions of the probability distributions for the single lives. In this section we use these expressions to solve probability problems and to obtain expectations, variances, and, for independent individual future lifetimes, the covariance of the joint-life and last-survivor future lifetimes.

Example 9.5.1

Assuming the future lifetimes of (80) and (85) are independent, obtain an expression, in single life table functions, for the probability that their

a. First death occurs after 5 and before 10 years from now, and
b. Last death occurs after 5 and before 10 years from now.

Solution:

a. With $T = T(80:85)$ we obtain

$$\begin{aligned}\Pr(5 < T \leq 10) &= \Pr(T > 5) - \Pr(T > 10) \\ &= {}_5p_{80:85} - {}_{10}p_{80:85} \\ &= {}_5p_{80}\, {}_5p_{85} - {}_{10}p_{80}\, {}_{10}p_{85}.\end{aligned}$$

Note that the independence assumption is used only in the last step.

b. With $T = T(\overline{80:85})$

$$\Pr(5 < T \leq 10) = \Pr(T > 5) - \Pr(T > 10)$$

$$= {}_5p_{\overline{80:85}} - {}_{10}p_{\overline{80:85}},$$

and from (9.4.6) we obtain

$$= {}_5p_{80} - {}_{10}p_{80} + {}_5p_{85} - {}_{10}p_{85} - ({}_5p_{80:85} - {}_{10}p_{80:85}).$$

Using the independence assumption, we can substitute ${}_5p_{80}\ {}_5p_{85}$ for ${}_5p_{80:85}$ and ${}_{10}p_{80}\ {}_{10}p_{85}$ for ${}_{10}p_{80:85}$. ▼

The results of Section 3.5 concerning expected values of the distribution of T, the time-until-the death of (x), are also valid if $T = T(u)$ the time-until-failure of a survival status (u). We used some of these ideas in the previous two sections; now we will state them explicitly.

From (3.5.2) we have that $\mathring{e}_u = \mathrm{E}[T(u)]$, which for a survival status (u) can be obtained from the formula

$$\mathring{e}_u = \int_0^\infty {}_tp_u \, dt. \tag{9.5.1}$$

If (u) is the joint-life status (xy), then

$$\mathring{e}_{xy} = \int_0^\infty {}_tp_{xy} \, dt, \tag{9.5.2}$$

and for the last-survivor status $(\overline{xy})$ we have

$$\mathring{e}_{\overline{xy}} = \int_0^\infty {}_tp_{\overline{xy}} \, dt. \tag{9.5.3}$$

Upon taking expectation of both sides of (9.4.1), we see that

$$\mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}. \tag{9.5.4}$$

From (3.5.5), the expected value of $K = K(u)$ is

$$e_u = \sum_{l}^{\infty} {}_kp_u$$

for a survival status, (u). Special cases include

$$e_{xy} = \sum_{l}^{\infty} {}_kp_{xy}$$

and

$$e_{\overline{xy}} = \sum_{l}^{\infty} {}_kp_{\overline{xy}}.$$

It follows from (9.4.11) that

$$e_{\overline{xy}} = e_x + e_y - e_{xy}. \tag{9.5.5}$$

The variance formulas derived in Section 3.5 can be used to calculate the variance of the future lifetime, or curtate-future-lifetime, of any survival status, (u). Thus,

$$\text{Var}[T(xy)] = 2\int_0^\infty {}_tp_{xy}\,dt - (\mathring{e}_{xy})^2 \tag{9.5.6}$$

and

$$\text{Var}[T(\overline{xy})] = 2\int_0^\infty {}_tp_{\overline{xy}}\,dt - (\mathring{e}_{\overline{xy}})^2. \tag{9.5.7}$$

In Example 9.2.1 we calculated the covariance of $T(x)$ and $T(y)$ for dependent future lifetimes. For the moment, we return to dependent future lifetimes to explore the covariance of $T(xy)$ and $T(\overline{xy})$ in the general case:

$$\text{Cov}[T(xy),\, T(\overline{xy})] = \text{E}[T(xy)\,T(\overline{xy})] - \text{E}[T(xy)]\,\text{E}[T(\overline{xy})]. \tag{9.5.8}$$

On the basis of (9.4.2),

$$\text{E}[T(xy)\,T(\overline{xy})] = \text{E}[T(x)\,T(y)].$$

Using this result and (9.5.4), we can write

$$\text{Cov}[T(xy),\, T(\overline{xy})] = \text{E}[T(x)\,T(y)] - \text{E}[T(xy)]\,\{\text{E}[T(x)] + \text{E}[T(y)] - \text{E}[T(xy)]\},$$

which can be rewritten as

$$= \text{Cov}[T(x),\, T(y)] + \{\text{E}[T(x)] - \text{E}[T(xy)]\}\,\{\text{E}[T(y)] - \text{E}[T(xy)]\}. \tag{9.5.9}$$

If $T(x)$ and $T(y)$ are uncorrelated, then

$$\text{Cov}[T(xy),\, T(\overline{xy})] = (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy}). \tag{9.5.10}$$

Since both factors of (9.5.10) must be non-negative, we can see that when $T(x)$ and $T(y)$ are uncorrelated, $T(xy)$ and $T(\overline{xy})$ are positively correlated, except in the trivial case where $\mathring{e}_x$ or $\mathring{e}_y$ equals $\mathring{e}_{xy}$.

Example 9.5.2

For $T(x)$ and $T(y)$ of Examples 9.2.1 and 9.2.3 determine (a) $\text{Cov}[T(xy),\, T(\overline{xy})]$, and (b) the correlation coefficient of $T(xy)$ and $T(\overline{xy})$.

Solution:

Most of the required calculations have been done in previous examples. The remaining calculations can be done readily with the p.d.f.'s that were determined there. We will complete the calculations by displaying intermediate results in tabular form along with the number of the formula being illustrated.

Item	Distribution	
	Examples 9.2.1, 9.3.1, 9.4.1	Examples 9.2.3, 9.3.2, 9.4.2
a. $\mathring{e}_x = \mathring{e}_y = \mathrm{E}[T(x)] = \mathrm{E}[T(y)]$	5	10/3
$\mathring{e}_{xy} = \mathrm{E}[T(xy)]$	2	2
$\mathrm{Cov}[T(x), T(y)]$	−25/3	0
$\mathrm{Cov}[T(xy), T(\overline{xy})]$	2/3	16/9
b. $\mathrm{E}[T(xy)^2]$	20/3	20/3
$\mathrm{Var}[T(xy)]$	8/3	8/3
$\mathrm{E}[T(\overline{xy})]$	8	14/3
$\mathrm{E}[T(\overline{xy})^2]$	200/3	80/3
$\mathrm{Var}[T(\overline{xy})]$	8/3	44/9
$\rho_{T(xy)\ T(\overline{xy})}$	1/4	$\sqrt{8/33}$

▼

9.6 Dependent Lifetime Models

In Section 9.2 the concept of the joint distribution of $[T(x), T(y)]$ was introduced. Two examples, which did not appear plausible as models for the joint distribution of $[T(x), T(y)]$ for human lives, were used to illustrate the ideas. Example 9.2.1 illustrated the ideas when $T(x)$ and $T(y)$ are dependent, and Example 9.2.3 provided similar illustrations for independent random variables.

In this section, two general approaches to specifying the joint distribution of $[T(x), T(y)]$ will be studied. The convenient independent case is included within each model.

9.6.1 Common Shock

Let $T^*(x)$ and $T^*(y)$ denote two future lifetime random variables that, in the absence of the possibility of a common shock, are independent; that is,

$$s_{T^*(x)\ T^*(y)}(s, t) = \Pr[T^*(x) > s \cap T^*(y) > t]$$

$$= s_{T^*(x)}(s)\ s_{T^*(y)}(t). \tag{9.6.1}$$

In addition, there is a ***common shock*** random variable, to be denoted by Z, that can affect the joint distribution of the time-until-death of lives (x) and (y). This

common shock random variable is independent of $[T^*(x), T^*(y)]$ and has an exponential distribution; that is,

$$s_Z(z) = e^{-\lambda z} \quad z > 0, \lambda \geq 0.$$

We can picture the random variable Z as being associated with the time of a catastrophe such as an earthquake or aircraft crash. The random variables of interest in building models for life insurance or annuities to (x) and (y) are $T(x) = \min[T^*(x), Z]$ and $T(y) = \min[T^*(y), Z]$. The joint survival function of $[T(x), T(y)]$ is

$$\begin{aligned} s_{T(x)T(y)}(s, t) &= \Pr\{\min[T^*(x), Z] > s \cap \min[T^*(y), Z] > t\} \\ &= \Pr\{[T^*(x) > s \cap Z > s] \cap [T^*(y) > t \cap Z > t]\} \\ &= \Pr[T^*(x) > s \cap T^*(y) > t \cap Z > \max(s, t)] \\ &= s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]}. \end{aligned} \tag{9.6.2}$$

The final line of (9.6.2) follows from the independence of $[T^*(x), T^*(y), Z]$.

Using the joint survival function displayed in (9.6.2), we can determine the joint p.d.f. of $[T(x), T(y)]$. Except when $s = t$, the p.d.f. can be found by partial differentiation. We have

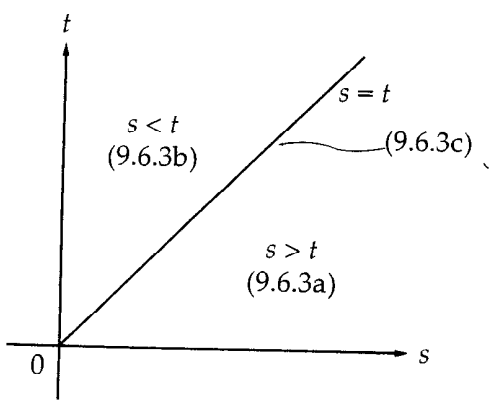
$$\begin{aligned} f_{T(x)T(y)}(s, t) &= \frac{\partial^2}{\partial s \partial t} s_{T^*(x)}(s)\, s_{T^*(y)}(t)\, e^{-\lambda[\max(s,t)]} \\ &= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s_{T^*(x)}(s)\, s'_{T^*(y)}(t)]\, e^{-\lambda s} \quad 0 < t < s \qquad (9.6.3a) \\ &= [s'_{T^*(x)}(s)\, s'_{T^*(y)}(t) - \lambda s'_{T^*(x)}(s)\, s_{T^*(y)}(t)]\, e^{-\lambda t} \quad 0 < s < t. \qquad (9.6.3b) \end{aligned}$$

This display does not complete the definition of the p.d.f. When $s = t$, the common shock contribution to the p.d.f. is

$$f_{T(x)T(y)}(t, t) = \lambda e^{-\lambda t}\, s_{T^*(x)}(t)\, s_{T^*(y)}(t) \qquad t \geq 0. \tag{9.6.3c}$$

The domain of this mixed p.d.f., with a ridge of density along the line $s = t$, is shown in Figure 9.6.1.

Domain of Common Shock p.d.f.



Remark:

The interpretation of the p.d.f. given in (9.6.3.a), (9.6.3.b), and (9.6.3c) requires a careful analysis of the distribution of $[T^*(x), T^*(y)]$ and the derived distribution of $\{T(x) = \min[T^*(x),Z],\ T(y) = \min[T^*(y), Z]\}$. Table 9.6.1 facilitates this analysis. Because of the intervention of the common shock, realizations of $T^*(x)$ and $T^*(y)$ may not be observed because of the prior realization of Z. The common shock random variable can mask or censor observations of $T^*(x)$ and $T^*(y)$. In addition, events such as $T^*(x) < Z < T^*(y)$ and $T^*(y) < Z < T^*(x)$ can contribute to the p.d.f.

The marginal survival functions are given by

$$s_{T(x)}(s) = \Pr\{[T(x) > s] \cap [T(y) > 0]\}$$
$$= s_{T^*(x)}(s)e^{-\lambda s} \tag{9.6.4a}$$

and

$$s_{T(y)}(t) = \Pr\{[T(x) > 0] \cap [T(y) > t]\}$$
$$= s_{T^*(x)}(t)e^{-\lambda t}. \tag{9.6.4b}$$

Interpretation of p.d.f. of $(T(x), T(y))$, Common Shock Model

Formula	Interpretation	Domain
9.6.3a	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > s)] \cup [(T^*(x) > s) \cap (t < T^*(y) \le t + \Delta t) \cap (s < Z \le s + \Delta s)]\} \approx [f_{T^*(x)}(s)\, f_{T^*(y)}(t)\, s_Z(s) + s_{T^*(x)}(s)\, f_{T^*(y)}(t)\, f_Z(s)]\Delta s \Delta t$	$0 < t < s$
9.6.3b	$\Pr\{[(s < T^*(x) \le s + \Delta s) \cap (t < T^*(y) \le t + \Delta t) \cap (Z > t)] \cup [(s < T^*(x) \le s + \Delta s) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)]\} \approx [f_{T^*(x)}(s)\, f_{T^*(y)}(t)\, s_Z(t) + f_{T^*(x)}(s)\, s_{T^*(y)}(t) f_Z(t)]\Delta s \Delta t$	$0 < s < t$
9.6.3c	$\Pr\{(T^*(x) > t) \cap (T^*(y) > t) \cap (t < Z \le t + \Delta t)\} \approx s_{T^*(x)}(t)\, s_{T^*(y)}(t)\, f_Z(t)\, \Delta t$	$s = t$

If we are interested in the distribution of $T(xy) = \min[T(x), T(y)]$, the joint-life status in the common shock model, the survival function can be obtained using (9.3.1) and (9.6.3):

$$s_{T(xy)}(t) = s_{T^*(x)}(t)\, s_{T^*(y)}(t)\, e^{-\lambda t} \quad 0 < t. \tag{9.6.5}$$

The distribution of $T(\overline{xy}) = \max[T(x), T(y)]$, the last-survivor status in the common shock model, can be derived by using (9.4.5), (9.6.5), (9.6.4a), and (9.6.4b):

$$s_{T(\overline{xy})}(t) = [s_{T^*(x)}(t) + s_{T^*(y)}(t) - s_{T^*(x)T^*(y)}(t, t)]\, e^{-\lambda t} \quad 0 < t. \tag{9.6.6}$$

If the common shock parameter $\lambda = 0$, formulas (9.6.5) and (9.6.6) revert to the form where $T(x)$ and $T(y)$ are independent. When $\lambda > 0$ the joint-life and last-survivor survival functions are each less than the corresponding survival function when $\lambda = 0$.

Example 9.6.1

Exhibit $\mu_{xy}(t)$ as derived from (9.6.5).

Solution:

$$\mu_{xy}(t) = -\frac{d}{dt}\log\left[s_{T^*(x)}(t)\, s_{T^*(y)}(t)e^{-\lambda t}\right]$$
$$= \mu(x+t) + \mu(y+t) + \lambda. \qquad \blacktriangledown$$

Example 9.6.2

The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameter μ_1, μ_2, and λ. These three random variables are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\, ds + f_{T(x)T(y)}(t, t) + \int_t^\infty f_{T(x)T(y)}(s, t)\, ds.$$

b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\, du.$$

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^\infty f_{T(x)T(y)}(t, t)\, dt.$$

Solution:

a. We use the three elements of (9.6.3), adapted for exponential distributions, to obtain

$$f_{T(y)}(t) = \int_0^t \mu_1(\mu_2+\lambda)e^{-\mu_1 s-(\mu_2+\lambda)t}\, ds + \lambda e^{-(\mu_1+\mu_2+\lambda)t}$$
$$+ \int_t^\infty \mu_2(\mu_1+\lambda)e^{-(\mu_1+\lambda)s-\mu_2 t}\, ds$$
$$= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t}(1 - e^{-\mu_1 t}) + \lambda e^{-(\mu_1+\mu_2+\lambda)t} + \mu_2 e^{-(\mu_1+\mu_2+\lambda)t}$$
$$= (\mu_2+\lambda)e^{-(\mu_2+\lambda)t} \quad 0 < t.$$

b. $s_{T(y)} = \int_t^\infty f_{T(y)}(u)\, du = e^{-(\mu_2+\lambda)t} = s_{T^*(y)}(t)e^{-\lambda t}$, which agrees with (9.6.4.b).

c. $\Pr[T(x) = T(y)] = \int_0^\infty \lambda e^{-(\mu_1+\mu_2+\lambda)t}\, dt = \dfrac{\lambda}{\lambda+\mu_1+\mu_2}$. $\blacktriangledown$

9.6.2 Copulas

The word ***copula*** means something that connects or joins together. Copula is used in multivariate statistical analysis to define a class of bivariate distributions with specified marginal distributions.

In this section we will illustrate actuarial applications of Frank's copula. The notation will be that used in Section 9.2. It is claimed that

$$F_{T(x)T(y)}(s, t) = \frac{1}{\alpha} \log \left[1 + \frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1} \right] \tag{9.6.7}$$

when $\alpha \neq 0$ is a joint d.f. with marginal distribution $F_{T(x)}(s)$ and $F_{T(y)}(t)$. This claim can be verified by confirming that

$$F_{T(x)T(y)}(0, 0) = 0, \tag{9.6.8}$$

$$F_{T(x)T(y)}(\infty, \infty) = 1, \tag{9.6.9}$$

$$\frac{\partial}{\partial s \partial t} F_{T(x)T(y)}(s, t) = f_{T(x)T(y)}(s, t)$$

$$= \frac{\alpha f_{T(x)}(s) f_{T(x)}(t)[e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]}]}{[(e^{\alpha} - 1) + (e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)]^2} (e^{\alpha} - 1) \geq 0, \tag{9.6.10}$$

and

$$F_{T(x)T(y)}(s, \infty) = F_{T(x)}(s),$$

$$F_{T(x)T(y)}(\infty, t) = F_{T(y)}(t).$$

Statements (9.6.8) and (9.6.9) are necessary for $F_{T(x)T(y)}(s, t)$ to be a d.f. of two future lifetime random variables. Statement (9.6.10) exhibits the joint p.d.f. and shows that it is non-negative.

The parameter α in the d.f. displayed in (9.6.7) and the p.d.f. displayed in (9.6.10) controls the dependence of $T(x)$ and $T(y)$. This can be appreciated by finding

$$\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s) f_{T(y)}(t) \left\{ \lim_{\alpha \to 0} [A(\alpha) B(\alpha) C(\alpha)] \right\},$$

where

$$A(\alpha) = e^{\alpha[F_{T(x)}(s)+F_{T(y)}(t)]},$$

$$B(\alpha) = \frac{(e^{\alpha} - 1)\alpha}{(e^{\alpha} - 1)^2},$$

and

$$C(\alpha) = \frac{1}{\{1 + [(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1) / (e^{\alpha} - 1)]\}^2}.$$

We have

$$\lim_{\alpha \to 0} A(\alpha) = 1,$$

$$\lim_{\alpha \to 0} B(\alpha) = 1,$$

and $\lim_{\alpha \to 0} C(\alpha)$ depends on the term in the denominator

$$\lim_{\alpha \to 0} \left[\frac{(e^{\alpha F_{T(x)}(s)} - 1)(e^{\alpha F_{T(y)}(t)} - 1)}{e^{\alpha} - 1} \right]$$

$$= \lim_{\alpha \to 0} \left\{ \frac{F_{T(x)}(s)\, e^{\alpha F_{T(x)}(s)}(e^{\alpha F_{T(y)}(t)} - 1) + F_{T(y)}(t) e^{\alpha F_{T(y)}(t)}(e^{\alpha F_{T(x)}(s)} - 1)}{e^{\alpha}} \right\}$$

$$= 0.$$

Therefore, $\lim_{\alpha \to 0} C(\alpha) = 1$, and $\lim_{\alpha \to 0} f_{T(x)T(y)}(s, t) = f_{T(x)}(s)\, f_{T(y)}(t)$. This means that $T(x)$ and $T(y)$ are independent in the limit as $\alpha \to 0$. We interpret the joint p.d.f. in this fashion when $\alpha = 0$.

Example 9.6.3

Let $F_{T(x)}(s) = s$, $0 < s \leq 1$, and $F_{T(y)}(t) = t$, $0 < t \leq 1$, in (9.6.7) and $T(xy) = \min[T(x), T(y)]$.

a. Find the d.f. of $T(xy)$.

b. Find the p.d.f. of $T(xy)$.

Solution:

a. Using (9.3.2),

$$F_{T(xy)}(t) = F_{T(x)}(t) + F_{T(y)}(t) - F_{T(x)T(y)}(t, t)$$

$$= 2t - \frac{1}{\alpha} \log \left[1 + \frac{(e^{\alpha t} - 1)^2}{e^{\alpha} - 1} \right] \qquad 0 < t < 1.$$

b. $$f_{T(xy)}(t) = \frac{d}{dt} F_{T(xy)}(t)$$

$$= 2 - \frac{2\alpha(e^{\alpha t} - 1)e^{\alpha t} / (e^{\alpha} - 1)}{\alpha\{1 + [(e^{\alpha t} - 1)^2 / (e^{\alpha} - 1)]\}}$$

$$= 2 - \left[\frac{2(e^{\alpha t} - 1)e^{\alpha t}}{(e^{\alpha} - 1) + (e^{\alpha t} - 1)^2} \right] \qquad 0 < t < 1.$$

▼

9.7 Insurance and Annuity Benefits

Insurances and annuities, previously discussed for an individual life, can be defined for the survival status, (u), and are discussed in this section. We also investigate more complicated examples in which an annuity, payable to a survival status, (u), is deferred until another status, (v), has failed.

9.7.1 Survival Statuses

With the single-life status, (x), replaced by the survival status, (u), the models and formulas of Chapters 4 and 5 are applicable here. Expressions for the actuarial

present values, variances, and percentiles in terms of the distribution of (u) are immediately available. The relationships of Sections 9.3 and 9.4 can then be used to be these expressions in terms of functions for the individual lives of the survival status.

For an insurance of unit amount payable at the end of the year in which the "survival" status fails, the model and formulas of Section 4.3 apply. Thus if K denotes the curtate-future-lifetime of (u), then the

- Time of payment is $K + 1$,
- Present value at issue of the payment is $Z = v^{K+1}$,
- Actuarial present value, A_u, is $\mathrm{E}[Z] = \sum_0^\infty v^{k+1} \Pr(K = k)$, (9.7.1)
- $\mathrm{Var}(Z) = {}^2A_u - (A_u)^2$. (9.7.2)

As an illustration, consider a unit sum insured payable at the end of the year in which the last survivor of (x) and (y) dies. From (9.4.16) and (9.7.1) we have

$$A_{\overline{xy}} = \sum_0^\infty v^{k+1} ({}_kp_x\, q_{x+k} + {}_kp_y\, q_{y+k} - {}_kp_{xy}\, q_{x+k:y+k}),$$

which can be used at forces of interest δ and 2δ to obtain the variance by (9.7.2).

The numerous formulas for discrete annuities in Section 5.3 are valid when the annuity payments are contingent on the survival of a survival status. For example, if we replace x with u to emphasize that K is the curtate-future-lifetime of the survival status, (u), we can restate the following formulas for an n-year temporary life annuity in regard to (u):

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & K = 0, 1, \ldots, n-1 \\ \ddot{a}_{\overline{n}|} & K = n, n+1, \ldots, \end{cases} \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n}|} = \mathrm{E}[Y] = \sum_0^{n-1} \ddot{a}_{\overline{K+1}|}\, {}_{k|}q_u + \ddot{a}_{\overline{n}|}\, {}_np_u, \qquad (9.7.3)$$

$$\ddot{a}_{u:\overline{n}|} = \sum_0^{n-1} {}_kE_u = \sum_0^{n-1} v^k\, {}_kp_u, \qquad \text{(5.3.9) restated}$$

$$\ddot{a}_{u:\overline{n}|} = \frac{1}{d}(1 - A_{u:\overline{n}|}), \qquad \text{(5.3.12) restated}$$

$$\mathrm{Var}(Y) = \frac{1}{d^2}[{}^2A_{u:\overline{n}|} - (A_{u:\overline{n}|})^2]. \qquad \text{(5.3.14) restated}$$

As an illustration, consider an annuity of 1, payable at the beginning of each year to which both (x) and (y) survive during the next n years. This is an annuity to the joint-life status (xy). By substituting ${}_tp_{xy}$, or ${}_tp_x\, {}_tp_y$ if the lifetimes are independent, for ${}_tp_u$ in the above formulas, we can obtain the actuarial present value of the annuity. For the variance as given in (5.3.14), we can use

$$A_{xy:\overline{n}|} = 1 - d\ddot{a}_{xy:\overline{n}|}$$

and

$$ {}^{2}A_{xy:\overline{n}|} = 1 - (2d - d^2)\, {}^{2}\ddot{a}_{xy:\overline{n}|}, $$

or we can calculate the actuarial present values directly.

In addition, we can establish relationships among the present-value random variables for annuities and insurances on the last-survivor status and the joint-life status. From relationship (9.4.13), we have

$$ v^{K(\overline{xy})+1} + v^{K(xy)+1} = v^{K(x)+1} + v^{K(y)+1}, \tag{9.7.4} $$

$$ \ddot{a}_{\overline{K(\overline{xy})+1}|} + \ddot{a}_{\overline{K(xy)+1}|} = \ddot{a}_{\overline{K(x)+1}|} + \ddot{a}_{\overline{K(y)+1}|}. \tag{9.7.5} $$

By taking the expectations of both sides of (9.7.4) and (9.7.5), we have

$$ A_{\overline{xy}} + A_{xy} = A_x + A_y $$

and

$$ \ddot{a}_{\overline{xy}} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y. $$

These formulas allow us to express the actuarial present values of last-survivor annuities and insurances in terms of those for the individual lives and the joint-life status. Note that these formulas hold for all joint distributions; independence is not required.

We now consider continuous insurances and annuities. If T, the future-lifetime random variable in Sections 4.2 and 5.2, is reinterpreted as $T(u)$, the time-until-failure of the survival status, (u), the formulas of those sections for present values, actuarial present values, percentiles, and variances hold for insurances and annuities for the status (u).

For an insurance paying a unit amount at the moment of failure of (u), the present value at policy issue, the actuarial present value and the variance are given by

$$ Z = v^T, $$

$$ \bar{A}_u = \int_0^\infty v^t\, {}_tp_u\, \mu_u(t)\, dt, \qquad \text{(4.2.6) restated} $$

$$ \mathrm{Var}(Z) = {}^{2}\bar{A}_u - (\bar{A}_u)^2. $$

As an illustration, the restated (4.2.6) for the last survivor of (x) and (y) would be

$$ \bar{A}_{\overline{xy}} = \int_0^\infty v^t\, {}_tp_{\overline{xy}}\, \mu_{\overline{xy}}(t)\, dt. $$

From (9.4.7), this is

$$ \bar{A}_{\overline{xy}} = \int_0^\infty v^t\, [{}_tp_x\, \mu(x+t) + {}_tp_y\, \mu(y+t) - {}_tp_{xy}\, \mu_{xy}(t)]\, dt. $$

For an annuity payable continuously at the rate of 1 per annum until the time-of-failure of (u), we have

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_u = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_tp_u\, \mu_u(t)\, dt \qquad \text{(5.2.3) restated}$$

$$= \int_0^\infty v^t\, {}_tp_u\, dt, \qquad \text{(5.2.4) restated}$$

$$\text{Var}(Y) = \frac{{}^2\bar{A}_u - (\bar{A}_u)^2}{\delta^2}. \qquad \text{(5.2.9) restated}$$

The interest identity,

$$\delta\, \bar{a}_{\overline{T}|} + v^T = 1, \qquad \text{(5.2.7) restated}$$

is also available for $T = T(u)$ and provides the connection between the models for insurances and annuities.

As an application, consider an annuity payable continuously at the rate of 1 per year as long as at least one of (x) or (y) survives. This is an annuity in respect to $(\overline{xy})$, so we have from the above formulas with $T = T(\overline{xy})$

$$Y = \bar{a}_{\overline{T}|},$$

$$\bar{a}_{\overline{xy}} = \int_0^\infty \bar{a}_{\overline{t}|}\, [{}_tp_x\, \mu(x+t) + {}_tp_y\, \mu(y+t) - {}_tp_{xy}\, \mu_{xy}(t)]\, dt$$

$$= \int_0^\infty v^t\, {}_tp_{\overline{xy}}\, dt,$$

$$\text{Var}(Y) = \frac{{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2}{\delta^2}.$$

Formula (9.4.3) implies that

$$v^{T(\overline{xy})} + v^{T(xy)} = v^{T(x)} + v^{T(y)} \qquad (9.7.6)$$

and

$$\bar{a}_{\overline{T(\overline{xy})}|} + \bar{a}_{\overline{T(xy)}|} = \bar{a}_{\overline{T(x)}|} + \bar{a}_{\overline{T(y)}|}, \qquad (9.7.7)$$

and (9.4.1) implies that

$$v^{T(\overline{xy})}\, v^{T(xy)} = v^{T(x)}\, v^{T(y)}. \qquad (9.7.8)$$

These identities can be used to obtain the relations among the actuarial present values, variances, and covariances of insurances and annuities for the various statuses. For example, taking the expectations of both sides of (9.7.6) and (9.7.7), we obtain

$$\bar{A}_{\overline{xy}} + \bar{A}_{xy} = \bar{A}_x + \bar{A}_y, \qquad (9.7.9)$$

$$\bar{a}_{\overline{xy}} + \bar{a}_{xy} = \bar{a}_x + \bar{a}_y. \qquad (9.7.10)$$

In the same way that $\text{Cov}[T(\overline{xy}), T(xy)]$ was expressed as

$$\text{Cov}[T(\overline{xy}), T(xy)] = \text{Cov}[T(x), T(y)] + (\bar{e}_x - \bar{e}_{xy})(\bar{e}_y - \bar{e}_{xy}),$$

it can be shown that

$$\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = \text{Cov}(v^{T(x)}, v^{T(y)}) + (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}). \qquad (9.7.11)$$

Both factors in the second term of (9.7.11) are nonpositive, so it will be non-negative and will be zero only in the trivial case where either $\bar{A}_x$ or $\bar{A}_y$ equals $\bar{A}_{xy}$.

The actuarial present value of a continuous annuity paid with respect to $(\overline{xy})$ where, because (x) and (y) are subject to common shock, the joint survival function is given by (9.6.5) can be written in current payment form as

$$\bar{a}_{\overline{xy}} = \int_0^\infty e^{-(\delta+\lambda)t} {}_t p_{T^*(x)}\, dt + \int_0^\infty e^{-(\delta+\lambda)t} {}_t p_{T^*(y)}\, dt$$

$$- \int_0^\infty e^{-(\delta+\lambda)t} {}_t p_{T^*(x)} {}_t p_{T^*(y)}\, dt. \qquad (9.7.12)$$

Formula (9.7.12) illustrates how the common shock parameter λ can be combined with the force of interest in some calculations.

Example 9.7.1

a. Extend (9.7.9) to the actuarial present value of an n-year term insurance paying a death benefit of 1 at the moment of the last death of (x) and (y) if this death occurs before time n. If at least one individual survives to time n, no payment is made.
b. Use the formula to calculate the actuarial present value, on the basis of $\delta = 0.05$, of a 5-year term insurance payable on the death of the last survivor of the two lives in Example 9.2.1.

Solution:

a. By restating (9.7.6) for n-year term insurance random variables and then taking expectation of both sides, we have

$$\bar{A}^{\,1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + \bar{A}^{\,1}_{y:\overline{n}|} - \bar{A}^{\,1}_{\overline{xy}\setminus:\overline{n}|}.$$

The symbol $\bar{A}^{\,1}_{\overline{xy}\setminus:\overline{n}|}$ represents the actuarial present value of an n-year term insurance payable at the failure of the joint-life status if it occurs prior to n.

b. From Example 9.2.1, $T(x)$ and $T(y)$ each has p.d.f. $f_T(t) = 0.0002\,[t^3 + (10 - t)^3]$, $0 \le t < 10$. Therefore,

$$\bar{A}^{\,1}_{x:\overline{5}|} = \bar{A}^{\,1}_{y:\overline{5}|} = \int_0^5 e^{-0.05t}\{0.0002[t^3 + (10 - t)^3]\}\, dt$$

$$= 0.4563.$$

From Example 9.3.3, we have that $f_{T(xy)}(t) = 0.0004(10 - t)^3$, $0 < t < 10$, so

$$\bar{A}^{\,1}_{\overline{xy}\setminus:\overline{5}|} = \int_0^5 e^{-0.05t}[0.0004(10 - t)^3]\, dt = 0.8614.$$

Using the result from part (a),

$$\bar{A}^{\,1}_{\overline{xy:5}|} = 2(0.4563) - 0.8614 = 0.0512.$$

▼

9.7.2 Special Two-Life Annuities

In this section we illustrate by an example special annuities in which the payment rate depends on the survival of two lives.

Example 9.7.2

An annuity is payable continuously at the rate of

- 1 per year while both (x) and (y) are alive,
- 2/3 per year while one of (x) or (y) is alive and the other is dead.

Derive expressions for

a. The annuity's present-value random variable
b. The annuity's actuarial present value
c. The variance of the random variable in (a), under the assumption that $T(x)$ and $T(y)$ are independent.

Solution:

a. The annuity is a combination of one that is payable at the rate of 2/3 per year while at least one of (x) and (y) is alive—until time $T(\overline{xy})$]—and one that is payable at the rate of 1/3 per year while both individuals are alive—until time $T(xy)$. The present value of the payments is

$$Z = \frac{2}{3}\,\bar{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\,\bar{a}_{\overline{T(xy)}|}.$$

b. The actuarial present value is

$$\mathrm{E}[Z] = \frac{2}{3}\,\bar{a}_{\overline{xy}} + \frac{1}{3}\,\bar{a}_{xy}.$$

Using (9.7.10) to substitute for $\bar{a}_{\overline{xy}}$, we have

$$\mathrm{E}[Z] = \frac{2}{3}\,\bar{a}_{x} + \frac{2}{3}\,\bar{a}_{y} - \frac{1}{3}\,\bar{a}_{xy}.$$

Alternatively, from (5.3.2B) restated, we have

$$\mathrm{E}[Z] = \frac{2}{3}\int_0^\infty v^t\,{}_tp_{\overline{xy}}\,dt + \frac{1}{3}\int_0^\infty v^t\,{}_tp_{xy}\,dt.$$

Then, by considering the three mutually exclusive cases as to which of the lives may be surviving when $(\overline{xy})$ is surviving at time t, we can write

$${}_tp_{\overline{xy}} = {}_tp_{xy} + ({}_tp_x - {}_tp_{xy}) + ({}_tp_y - {}_tp_{xy}).$$

Substitution of this expression into the first integral and combining the results with the second give

$$E[Z] = \int_0^\infty v^t \, {}_tp_{xy} \, dt + \frac{2}{3}\int_0^\infty v^t \, ({}_tp_x - {}_tp_{xy}) \, dt$$
$$+ \frac{2}{3}\int_0^\infty v^t \, ({}_tp_y - {}_tp_{xy}) \, dt.$$

This expression of $E[Z]$ can be directly obtained by considering the three cases. The first term is the actuarial present value of the payments at the rate of 1 per year while both (x) and (y) survive. The second term is the actuarial present value of the payments at the rate of $2/3$ per year at those times t when (x) is alive (with probability ${}_tp_x$) but not both of (x) and (y) are alive (with probability ${}_tp_{xy}$). The third term has a similar interpretation with x and y interchanged.

c.
$$\text{Var}(Z) = \text{Var}\left(\frac{2}{3}\bar{a}_{\overline{T(\overline{xy})}|} + \frac{1}{3}\bar{a}_{\overline{T(xy)}|}\right)$$
$$= \frac{4}{9}\text{Var}(\bar{a}_{\overline{T(\overline{xy})}|}) + \frac{1}{9}\text{Var}(\bar{a}_{\overline{T(xy)}|}) + \frac{4}{9}\text{Cov}(\bar{a}_{\overline{T(\overline{xy})}|}, \bar{a}_{\overline{T(xy)}|}).$$

But, by Exercise 9.23, for independent $T(x)$ and $T(y)$,

$$\text{Cov}(\bar{a}_{\overline{T(\overline{xy})}|}, \bar{a}_{\overline{T(xy)}|}) = \frac{\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)})}{\delta^2}$$
$$= \frac{(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}.$$

Hence

$$\text{Var}(Z) = \left\{\frac{4/9\,[{}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2] + 1/9\,[{}^2\bar{A}_{xy} - (\bar{A}_{xy})^2] + 4/9\,(\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})}{\delta^2}\right\}.$$

▼

9.7.3 Reversionary Annuities

A ***reversionary annuity*** is payable during the existence of a status (u), but only after the failure of a second status (v). Conceptually, this is a deferred life annuity with a random deferment period equal to the time until failure of the second status. In fact, it is a generalization of the deferred life annuity, for if (v) is an n-year term certain, then reversionary annuity reduces to an n-year deferred annuity. If (u) is a term certain, the reversionary annuity reduces to a form of insurance, family income insurance, studied in Chapter 17. The basic notation for the actuarial present value of this annuity is $a_{v|u}$ with adornments to indicate frequency and timing of payments. The concept has been useful to obtain expressions for the more complex annuity arrangements in terms of single and joint status annuities (see Example 9.7.3). Here we will study the present-value random variables for reversionary annuities.

We start with an annuity of 1 per year payable continuously to (y) after the death of (x). The present value at 0, denoted by Z, is

$$Z = \begin{cases} {}_{T(x)|}\bar{a}_{\overline{T(y)-T(x)|}} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases} \tag{9.7.13}$$

This can be written as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(x)|}} & T(x) \le T(y) \\ 0 & T(x) > T(y), \end{cases} \tag{9.7.14}$$

or as

$$Z = \begin{cases} \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(x)|}} & T(x) \le T(y) \\ \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(y)|}} & T(x) > T(y), \end{cases} \tag{9.7.15}$$

which is the same as

$$Z = \bar{a}_{\overline{T(y)|}} - \bar{a}_{\overline{T(xy)|}}. \tag{9.7.16}$$

Thus,

$$\bar{a}_{x|y} = \mathrm{E}[Z] = \mathrm{E}[\bar{a}_{\overline{T(y)|}}] - \mathrm{E}[\bar{a}_{\overline{T(xy)|}}] = \bar{a}_y - \bar{a}_{xy}. \tag{9.7.17}$$

Formulas (9.7.16) and (9.7.17) hold for survival statuses (u) and (v). For example,

$$\bar{a}_{x:\overline{n}|y} = \bar{a}_y - \bar{a}_{xy:\overline{n}|},$$

$$\bar{a}_{x|y:\overline{n}|} = \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}.$$

We note that (9.7.17) holds for dependent future lifetimes.

Example 9.7.3

Calculate the actuarial present value of an annuity payable continuously at the rates shown in the following display.

Case, Rate, and Condition

1. 1 per year with certainty until time n,
2. 1 per year after time n if both (x) and (y) are alive,
3. 3/4 per year after time n if (x) is alive and (y) is dead, and
4. 1/2 per year after time n if (y) is alive and (x) is dead.

Solution:

We use the reversionary annuity idea to write the actuarial present value of this arrangement in terms of single-life and joint-life annuities.

- Case 1: This is an n-year annuity-certain: $\bar{a}_{\overline{n}|}$.
- Case 2: This is n-year deferred joint life annuity to (xy):

$$ {}_{n|}\bar{a}_{xy} = \bar{a}_{xy} - \bar{a}_{xy:\overline{n}|}.$$

- Case 3: This is a reversionary annuity of 3/4 per annum to x after $\overline{y:\overline{n}|}$:

$$\frac{3}{4}\,\bar{a}_{\overline{y:\overline{n}|}x} = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{x:(\overline{y:\overline{n}|})}) = \frac{3}{4}\,(\bar{a}_x - \bar{a}_{xy} - \bar{a}_{x:\overline{n}|} + \bar{a}_{xy:\overline{n}|}).$$

- Case 4: This is a reversionary annuity of 1/2 per annum to y after $\overline{x:\overline{n}|}$:

$$\frac{1}{2}\,\bar{a}_{\overline{x:\overline{n}|}y} = \frac{1}{2}\,(\bar{a}_y - \bar{a}_{xy} - \bar{a}_{y:\overline{n}|} + \bar{a}_{xy:\overline{n}|}).$$

Adding the results for the four cases together, we obtain the required actuarial present value,

$$\bar{a}_{\overline{n}|} + \frac{3}{4}\,\bar{a}_x + \frac{1}{2}\,\bar{a}_y - \frac{1}{4}\,\bar{a}_{xy} - \frac{3}{4}\,\bar{a}_{x:\overline{n}|} - \frac{1}{2}\,\bar{a}_{y:\overline{n}|} + \frac{1}{4}\,\bar{a}_{xy:\overline{n}|}. \qquad \blacktriangledown$$

9.8 Evaluation—Special Mortality Assumptions

In Section 9.7 the actuarial present values of a variety of insurance and annuity benefits that involve two lifetime random variables were developed. These developments typically culminated in an integral or summation. In this section we study several assumptions about the distribution of $T(u)$ that will simplify the evaluation of these integrals and summations.

9.8.1 Gompertz and Makeham Laws

Here we examine the assumption that mortality follows Makeham's law, or its important special case, Gompertz's law, and the implications for the computations of actuarial present values with respect to multiple life statuses. Independent future lifetime random variables will be assumed.

We begin with the assumption that mortality for each life follows Gompertz's law, $\mu(x) = Bc^x$. We seek to substitute a single-life survival status (w) that has a force of failure equal to the force of failure of the joint-life status (xy) for all $t \geq 0$. Consider

$$\mu_{xy}(s) = \mu(w + s) \qquad s \geq 0; \tag{9.8.1}$$

that is,

$$Bc^{x+s} + Bc^{y+s} = Bc^{w+s},$$

or

$$c^x + c^y = c^w, \tag{9.8.2}$$

which defines the desired w. It follows that for $t > 0$,

$$\begin{aligned} {}_tp_w &= \exp\left[-\int_0^t \mu(w + s)\,ds\right] \\ &= \exp\left[-\int_0^t \mu_{xy}(s)\,ds\right] \\ &= {}_tp_{xy}. \end{aligned} \tag{9.8.3}$$

Thus for w defined in (9.8.2), all probabilities, expected values, and variances for the joint-life status (xy) equal those for the single life (w). For tabled values, the

need for a two-dimensional array has been replaced by the need for a one-dimensional array, but typically w is nonintegral, and therefore the determination of its values requires interpolation in the single array.

The assumption that mortality for each life follows Makeham's law (see Table 3.6) makes the search more complex. The force of mortality for the joint-life status is

$$\mu_{xy}(s) = \mu(x + s) + \mu(y + s) = 2A + Bc^s (c^x + c^y). \tag{9.8.4}$$

We cannot substitute a single life for the two lives because of the $2A$. Instead, we replace (xy) with another joint-life status (ww), and then

$$\mu_{ww}(s) = 2\mu(w + s) = 2(A + Bc^s c^w), \tag{9.8.5}$$

and we choose w such that

$$2c^w = c^x + c^y. \tag{9.8.6}$$

Unlike the Gompertz case where the one-dimensional array is based on functions from a single life table, this one-dimensional array is based on functions for a joint-life status (ww) involving equal-age lives.

Example 9.8.1

Use (3.7.1) and the $\ddot{a}_{xx}$ values based on the Illustrative Life Table (Appendix 2A) with interest at 6% to calculate the value of $\ddot{a}_{60:70}$. Compare your result with the values of $\ddot{a}_{60:70}$ in the table of $\ddot{a}_{x:x+10}$.

Solution:

From $c = 10^{0.04}$ and $c^{60} + c^{70} = 2c^w$, we obtain $w = 66.11276$. Then using linear interpolation, $\ddot{a}_{60:70} = 0.88724\ddot{a}_{66:66} + 0.11276\ddot{a}_{67:67} = 7.55637$. The value by the $\ddot{a}_{x:x+10}$ table is 7.55633. ▼

9.8.2 Uniform Distribution

We retain the independence assumption, and in addition we assume a uniform distribution of deaths in each year of age for each individual in the joint-life status. With this additional assumption, we can evaluate the actuarial present values of annuities payable more frequently than once a year and insurance benefits payable at the moment of death.

We recall from Table 3.6 that, under the assumption of a uniform distribution of deaths for each year of age, ${}_tp_x = 1 - tq_x$ and

$${}_tp_x\, \mu(x + t) = \frac{d}{dt}(1 - {}_tp_x) = q_x. \tag{9.8.7}$$

When we apply this assumption to a joint-life status (xy), with independent $T(x)$ and $T(y)$, we obtain, for $0 \le t \le 1$,

$$
\begin{aligned}
{}_tp_{xy}\,\mu_{xy}(t) &= {}_tp_x\,{}_tp_y\,[\mu(x+t)+\mu(y+t)] \\
&= {}_tp_y\,{}_tp_x\,\mu(x+t) + {}_tp_x\,{}_tp_y\,\mu(y+t) \\
&= (1-tq_y)q_x + (1-tq_x)q_y \\
&= q_x + q_y - q_xq_y + (1-2t)q_xq_y \\
&= q_{xy} + (1-2t)q_xq_y. \qquad (9.8.8)
\end{aligned}
$$

On the basis of (4.4.1), the actuarial present value for an insurance benefit in regard to a survival status, (u), can be written as

$$
\bar{A}_u = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_u \int_0^1 (1+i)^{1-s}\,\frac{{}_{k+s}p_u}{{}_kp_u}\,\mu_u(k+s)\,ds.
$$

Using (9.8.8), we can rewrite this for the joint-life status, (xy), as

$$
\begin{aligned}
\bar{A}_{xy} &= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\left[q_{x+k:y+k}\int_0^1 (1+i)^{1-s}\,ds\right. \\
&\qquad \left. + q_{x+k}\,q_{y+k}\int_0^1 (1+i)^{1-s}\,(1-2s)\,ds\right] \\
&= \frac{i}{\delta}A_{xy} + \frac{i}{\delta}\left(1-\frac{2}{\delta}+\frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \qquad (9.8.9)
\end{aligned}
$$

To interpret the right-hand side of (9.8.9), we see from (4.4.2) that the first term is equal to $\bar{A}_{xy}$ if $T(xy)$, the time-until-failure of (xy), is uniformly distributed in each year of future lifetime. Such is not the case for $T(xy) = \{\min[T(x), T(y)]\}$ when $T(x)$ and $T(y)$ are distributed independently and uniformly over such years. Under this latter assumption, the conditional distribution of $T(xy)$, given that $T(x)$ and $T(y)$ are in different yearly intervals, is also uniform over each year of future lifetime. However, given that $T(x)$ and $T(y)$ are within the same interval, the distribution of their minimum is shifted toward the beginning of the interval (see Exercise 9.38). A consequence of this shift is to require the second term in (9.8.9) to cover the additional expected costs of the earlier claims in those years. The second term, which is the product of an interest term that is close to $i/6$ (see Exercise 9.39) and a actuarial present value for an insurance payable if both individuals die in the same future year, is very small. The actuarial present value $\bar{A}_{xy}$ is often approximated by ignoring the small correction term, thereby simplifying (9.8.9) to

$$
\bar{A}_{xy} \cong \frac{i}{\delta}A_{xy}, \qquad (9.8.10)
$$

which is exact, as noted previously, if $T(xy)$ is uniformly distributed in each year of future lifetime.

To evaluate $\bar{a}_{xy}$, we have from (5.2.8), with survival status (x) replaced by (xy),

$$
\bar{a}_{xy} = \frac{1}{\delta}(1-\bar{A}_{xy}),
$$

and, on substitution from (9.8.9), obtain

$$\bar{a}_{xy} = \frac{1}{\delta}\left\{1 - \frac{i}{\delta}\left[A_{xy} + \left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}\right]\right\}.$$

Now, on the basis of (5.3.7) for the status (xy), we substitute $1 - d\ddot{a}_{xy}$ for A_{xy} and use (5.4.12) and (5.4.13) to write

$$\bar{a}_{xy} = [\alpha(\infty)\ddot{a}_{xy} - \beta(\infty)] - \frac{i}{\delta^2}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \tag{9.8.11}$$

Formula (9.8.11) follows from the assumption that $T(x)$ and $T(y)$ are distributed independently and uniformly over future years. If we assume that $T(xy)$ itself is uniformly distributed over each future year, then from the continuous case of (5.4.11), $m = \infty$, we would have immediately

$$\bar{a}_{xy} = \alpha(\infty)\ddot{a}_{xy} - \beta(\infty). \tag{9.8.12}$$

Formula (9.8.12) differs from (9.8.11) by a small amount, which approximates the product of $i/(6\delta)$ and the actuarial present value for an insurance payable if both individuals die in the same future year.

To use the same approach to evaluate the actuarial present value of an annuity-due payable m-thly, we need an expression for $A_{xy}^{(m)}$ under the assumption of a uniform distribution of deaths for each of the individuals in each year of age. In analogy to the continuous case, we start with

$$A_{xy}^{(m)} = \sum_{k=0}^{\infty} v^k\,{}_kp_{xy}\sum_{j=1}^{m} v^{j/m}\,({}_{(j-1)/m}p_{x+k:y+k} - {}_{j/m}p_{x+k:y+k}). \tag{9.8.13}$$

In Exercise 9.40 this expression, under the assumption that $T(x)$ and $T(y)$ are independently and uniformly distributed over each year of age, is reduced to

$$A_{xy}^{(m)} = \frac{i}{i^{(m)}}A_{xy} + \frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \tag{9.8.14}$$

As $m \to \infty$, the expression in (9.8.14) approach their counterparts in (9.8.9). To interpret the right-hand side of (9.8.14), we see by analogy to (9.8.9) that the first term is the usual approximation for $A_{xy}^{(m)}$ and is exact if $T(xy)$ is uniformly distributed in each year. Then,

$$\frac{i}{i^{(m)}}\left(1 + \frac{1}{m} - \frac{2}{d^{(m)}} + \frac{2}{i}\right) \cong \frac{m^2 - 1}{6m^2}\,i,$$

which is less than $i/6$.

By substituting (9.8.14) into ~~(5.4.4)~~ restated for (xy), and replacing A_{xy} by $1 - d\ddot{a}_{xy}$, we obtain the formula for $\ddot{a}_{xy}^{(m)}$ that is analogous to (9.8.11). If the second term of (9.8.14) is ignored, the formula for $\ddot{a}_{xy}^{(m)}$ reduces to

$$\ddot{a}_{xy}^{(m)} = \alpha(m)\ddot{a}_{xy} - \beta(m). \tag{9.8.15}$$

Again by (5.4.11), this is exact under the assumption that the distribution of $T(xy)$ is uniform over each year of future lifetime.

9.9 Simple Contingent Functions

In this section we study insurances that, in addition to being dependent on the time of failure of the status, are contingent on the order of the deaths of the individuals in the group. In this section we will assume that $T(x)$ and $T(y)$ have a continuous joint d.f. This is done to exclude the common shock model of Section 9.6.1.

We begin with an evaluation of the probability that (x) dies before (y) and before n years from now. In IAN this probability is denoted by ${}_nq^1_{xy}$, where the 1 over the x indicates the probability is for an event in which (x) dies before (y), and n indicates that the event occurs within n years. Then ${}_nq^1_{xy}$ equals the double integral of the joint p.d.f. of $T(x)$ and $T(y)$ over the set of outcomes such that $T(x) \le T(y)$ and $T(x) \le n$:

$$
\begin{aligned}
{}_nq^1_{xy} &= \int_0^n \int_s^\infty f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^\infty f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s|T(x) = s]\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[T(y) > s|T(x) = s]\, {}_sp_x\, \mu(x + s)\, ds. \qquad (9.9.1)
\end{aligned}
$$

For the independent case, $\Pr[T(y) \ge s|T(x) = s] = {}_sp_y$, so

$$
{}_nq^1_{xy} = \int_0^n {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \qquad (9.9.2)
$$

An interpretation of (9.9.2) involves three elements. First, because s is the time of death of (x), the probability ${}_sp_x\, {}_sp_y$ indicates that both (x) and (y) survive to time s. Second, $\mu(x + s)\, ds$ is the probability that (x), now age $x + s$, will die in the interval $(s, s + ds)$. Third, the probabilities are summed for all times s between 0 and n.

Example 9.9.1

Calculate ${}_5q^1_{xy}$ for the lives in Example 9.2.1.

Solution:

From (9.9.1),

$$
\begin{aligned}
{}_5q^1_{xy} &= \int_0^5 \int_s^{10} 0.0006(t - s)^2\, dt\, ds \\
&= \int_0^5 0.0002(10 - s)^3\, ds = 0.46875.
\end{aligned}
$$

▼

We can also evaluate the probability that (y) dies after (x) and before n years from now. This probability is denoted by ${}_nq^2_{xy}$, the 2 above the y indicating that (y) dies second and n requiring that this occurs within n years. To evaluate ${}_nq^2_{xy}$, we integrate the joint p.d.f. of $T(x)$ and $T(y)$ over the event $[0 \le T(x) \le T(y) \le n]$:

$$
\begin{aligned}
{}_nq^2_{xy} &= \int_0^n \int_0^t f_{T(x)T(y)}(s, t)\, ds\, dt \\
&= \int_0^n \int_0^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt \\
&= \int_0^n \Pr[T(x) \le t | T(y) = t]\, {}_tp_y\, \mu(y + t)\, dt. \qquad (9.9.3)
\end{aligned}
$$

Again in the independent case, we have

$$
\begin{aligned}
{}_nq^2_{xy} &= \int_0^n {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\
&= {}_nq_y - {}_nq^1_{xy} \qquad (9.9.4)
\end{aligned}
$$

If the integration of (9.9.3) is set up in the reverse order, we have

$$
\begin{aligned}
{}_nq^2_{xy} &= \int_0^n \int_s^n f_{T(x)T(y)}(s, t)\, dt\, ds \\
&= \int_0^n \int_s^n f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\
&= \int_0^n \Pr[s < T(y) \le n | T(x) = s]\, {}_sp_x\, \mu(x + s)\, ds.
\end{aligned}
$$

Making the assumption of independence for $T(x)$ and $T(y)$, we can rewrite this as

$$
\begin{aligned}
{}_nq^2_{xy} &= \int_0^n ({}_sp_y - {}_np_y)\, {}_sp_x\, \mu(x + s)\, ds \\
&= {}_nq^1_{xy} - {}_np_y\, {}_nq_x. \qquad (9.9.5)
\end{aligned}
$$

In (9.9.5) the integrand is interpreted as the probability that (x) dies at time s, with $0 < s < n$, and (y) survives to time s but not to time n. Moreover, we now have that

$${}_nq^1_{xy} = {}_nq^2_{xy} + {}_np_y\, {}_nq_x.$$

This implies

$${}_nq^1_{xy} \ge {}_nq^2_{xy}.$$

Similar integrals can be written for the actuarial present values of contingent insurances, but some do not simplify to the same extent. Consider the actuarial present value of an insurance of 1 payable at the moment of death of (x) provided that (y) is still alive. This actuarial present value, denoted by $\bar{A}^1_{xy}$, is E[Z] where

$$Z = \begin{cases} v^{T(x)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Since Z is a function of $T(x)$ and $T(y)$, the expectation of Z can be obtained by using the joint p.d.f. of $T(x)$ and $T(y)$:

$$\begin{aligned} \bar{A}^1_{xy} &= \int_0^\infty \int_s^\infty v^s f_{T(x)T(y)}(s, t)\, dt\, ds \\ &= \int_0^\infty \int_s^\infty v^s f_{T(y)|T(x)}(t|s)\, dt\, f_{T(x)}(s)\, ds \\ &= \int_0^\infty \left[\int_s^\infty F_{T(y)|T(x)}(t|s)\, dt \right] v^s\, {}_sp_x\, \mu(x+s)\, ds. \end{aligned} \tag{9.9.6}$$

In the case of independent future lifetimes, $T(x)$ and $T(y)$, we can simplify (9.9.6) and express it in IAN as

$$\begin{aligned} \bar{A}^1_{xy} &= \int_0^\infty \left[\int_s^\infty {}_tp_y\, \mu(y+t)\, dt \right] v^s\, {}_sp_x\, \mu(x+s)\, ds \\ &= \int_0^\infty v^s\, {}_sp_y\, {}_sp_x\, \mu(x+s)\, ds. \end{aligned} \tag{9.9.7}$$

The final expression can be interpreted as follows: If (x) dies at any future time s and (y) is still surviving, then a payment of 1, with present value v^s, is made. When $\delta = 0$, $\bar{A}^1_{xy} = {}_\infty q^1_{xy}$.

Example 9.9.2

Determine the actuarial present value of a payment of 1,000 at the moment of death of (x) providing that (y) is still alive for (x) and (y) in Example 9.2.3 and on the basis of $\delta = 0.04$.

Solution:

Since $T(x)$ and $T(y)$ are independent in Example 9.2.3, we can use the results of that example in (9.9.7) to have

$$\begin{aligned} 1{,}000\, \bar{A}^1_{xy} &= 1{,}000 \int_0^\infty e^{-0.04s}\, {}_sp_y\, {}_sp_x\, \mu(x+s)\, ds \\ &= 1{,}000 \int_0^{10} e^{-0.04s}\, 0.01(10-s)^2\, 0.02(10-s)\, ds \\ &= 0.2 \int_0^{10} e^{-0.04s} (10-s)^3\, ds = 462.52. \end{aligned}$$

▼

Example 9.9.3

a. Derive the single integral expression for the actuarial present value of an insurance of 1 payable at the time of death of (y) if predeceased by (x).
b. Simplify the integral under the assumption of independent future lifetimes.

c. Obtain a second answer for part (b) by reversing the order of integration in the part (b) double integral.

Solution:

a. The actuarial present value, denoted by $\bar{A}^2_{xy}$, is E[Z] where

$$Z = \begin{cases} v^{T(y)} & T(x) \le T(y) \\ 0 & T(x) > T(y). \end{cases}$$

Again, Z is a function of $T(x)$ and $T(y)$, so we write an integral for the expectation of Z by using the joint p.d.f. of $T(x)$ and $T(y)$,

$$\begin{aligned} \bar{A}^2_{xy} &= \int_0^\infty \int_0^t v^t f_{T(x)T(y)}(s, t)\, ds\, dt \\ &= \int_0^\infty v^t f_{T(x)|T(y)}(s|t)\, ds\, f_{T(y)}(t)\, dt \\ &= \int_0^\infty v^t \Pr[T(x) \le t | T(y) = t]\, f_{T(y)}(t)\, dt. \end{aligned}$$

b. Invoking the independence assumption and writing in IAN we have

$$\begin{aligned} \bar{A}^2_{xy} &= \int_0^\infty v^t\, {}_tq_x\, {}_tp_y\, \mu(y + t)\, dt \\ &= \int_0^\infty v^t (1 - {}_tp_x)\, {}_tp_y\, \mu(y + t)\, dt \\ &= \bar{A}_y - \bar{A}^1_{xy}. \end{aligned}$$

We note for the independent case that we can express the actuarial present value for a simple contingent insurance, payable on a death other than the first death, in terms of the actuarial present values for insurances payable on the first death. This is the initial step in the numerical evaluation of simple contingent insurances for independent lives.

c. We have

$$\bar{A}^2_{xy} = \int_0^\infty \int_s^\infty v^t\, {}_sp_x\, \mu(x + s)\, {}_tp_y\, \mu(y + t)\, dt\, ds.$$

To simplify we replace t with $r + s$ in the inner integral and rewrite the expression

$$\begin{aligned} \bar{A}^2_{xy} &= \int_0^\infty \int_0^\infty v^{r+s}\, {}_{r+s}p_y\, \mu(y + r + s)\, {}_sp_x\, \mu(x + s)\, dr\, ds \\ &= \int_0^\infty v^s\, {}_sp_y\, {}_sp_x\, \mu(x + s) \left[\int_0^\infty v^r\, {}_rp_{y+s}\, \mu(y + s + r)\, dr \right] ds \\ &= \int_0^\infty v^s\, \bar{A}_{y+s}\, {}_sp_y\, {}_sp_x\, \mu(x + s)\, ds. \end{aligned}$$

This last integral is an application of the general result given in (2.2.10), E[W] = E[E[$W|V$]]. Here $V = T(x)$, $W = Z$, and we see that the conditional

expectation of Z, given $T(x) = s$, is the actuarial present value, $v^s\,{}_sp_y\,\bar{A}_{y+s}$, of the pure endowment for an amount A_{y+s} sufficient to fund a unit insurance on $(y + s)$. ▼

9.10 Evaluation—Simple Contingent Functions

We now turn to the evaluation of simple contingent probabilities and actuarial present values, noting the effects of assuming Gompertz's law, Makeham's law, and a uniform distribution of deaths. The ubiquitous assumption of independence will be made.

Example 9.10.1

Assuming Gompertz's law for the forces of mortality, calculate

a. The actuarial present value for an n-year term contingent insurance paying a unit amount at the moment of death of (x) only if (x) dies before (y)
b. The probability that (x) dies within n years and predeceases (y).

Solution:

a.
$$\bar{A}^1_{xy:\overline{n}|} = \int_0^n v^t\,{}_tp_{xy}\,\mu(x + t)\,dt.$$

Under Gompertz's law,

$$\begin{aligned}\bar{A}^1_{xy:\overline{n}|} &= \int_0^n v^t\,{}_tp_{xy}\,Bc^x\,c^t\,dt \\ &= \frac{c^x}{c^x + c^y}\int_0^n v^t\,{}_tp_{xy}\,B(c^x + c^y)c^t\,dt \\ &= \frac{c^x}{c^x + c^y}\int_0^n v^t\,{}_tp_{xy}\,\mu_{xy}(t)\,dt \\ &= \frac{c^x}{c^x + c^y}\,\bar{A}^{\,1}_{\overline{xy}\backslash:\overline{n}|}. \qquad (9.10.1)\end{aligned}$$

Furthermore, if (9.8.2) holds, then

$$\bar{A}^{\,1}_{\overline{xy}\backslash:\overline{n}|} = \bar{A}^1_{w:\overline{n}|},$$

and

$$\bar{A}^1_{xy:\overline{n}|} = \frac{c^x}{c^w}\,\bar{A}^1_{w:\overline{n}|}. \qquad (9.10.2)$$

b. Referring to (9.9.2) we see that ${}_nq^1_{xy}$ is $\bar{A}^1_{xy:\overline{n}|}$ with $v = 1$. Thus, it follows from (9.10.2) that, under Gompertz's law,

$${}_nq^1_{xy} = \frac{c^x}{c^w}\,{}_nq_w, \qquad (9.10.3)$$

where $c^w = c^x + c^y$. ▼

Example 9.10.2

Assuming Makeham's law for the forces of mortality, repeat Example 9.10.1.

Solution:

a. $$\bar{A}^1_{xy:\overline{n}|} = \int_0^n v^t \, {}_tp_{xy} \, (A + Bc^x c^t) \, dt$$

$$= A \int_0^n v^t \, {}_tp_{xy} \, dt + \frac{c^x}{c^x + c^y} \int_0^n v^t \, {}_tp_{xy} \, B(c^x + c^y)c^t \, dt$$

$$= A \left(1 - \frac{2c^x}{c^x + c^y}\right) \int_0^n v^t \, {}_tp_{xy} \, dt$$

$$+ \frac{c^x}{c^x + c^y} \int_0^n v^t \, {}_tp_{xy} \, [2A + B(c^x + c^y)c^t] \, dt$$

$$= A \left(1 - \frac{2c^x}{c^x + c^y}\right) \bar{a}_{xy:\overline{n}|} + \frac{c^x}{c^x + c^y} \bar{A}^{\,1}_{\overline{xy}\backslash:\overline{n}|}.$$

Then by using (9.8.6), we obtain

$$\bar{A}^1_{xy:\overline{n}|} = A \left(1 - \frac{c^x}{c^w}\right) \bar{a}_{ww:\overline{n}|} + \frac{c^x}{2c^w} \bar{A}^{\,1}_{\overline{ww}\backslash:\overline{n}|}. \tag{9.10.4}$$

b. Again, we set $v = 1$ in the result of part (a) to have

$${}_nq^1_{xy} = A \left(1 - \frac{c^x}{c^w}\right) \mathring{e}_{ww:\overline{n}|} + \frac{c^x}{2c^w} \, {}_nq_{ww}. \tag{9.10.5}$$

▼

The actuarial present value for a contingent insurance payable at the end of the year of death is

$$A^1_{xy} = \sum_{k=0}^{\infty} v^{k+1} \, {}_kp_{xy} \, q^{\,1}_{\overline{x+k:y+k}}. \tag{9.10.6}$$

Under the assumption of a uniform distribution of deaths for each individual and independence between the pair of future lifetime random variables, we have

$$q^{\,1}_{\overline{x+k:y+k}} = \int_0^1 {}_sp_{x+k:y+k} \, \mu(x + k + s) \, ds$$

$$= \int_0^1 q_{x+k}(1 - sq_{y+k}) \, ds$$

$$= q_{x+k} \left(1 - \frac{1}{2} q_{y+k}\right). \tag{9.10.7}$$

We can now rewrite ${}_sp_{x+k:y+k} \, \mu(x + k + s)$ in terms of $q^{\,1}_{\overline{x+k:y+k}}$,

$$
\begin{aligned}
{}_sp_{x+k:y+k}\,\mu(x+k+s) &= q_{x+k}(1-sq_{y+k}) \\
&= q_{x+k}\left(1-\frac{1}{2}\,q_{y+k}\right) \\
&\quad + \left(\frac{1}{2}-s\right) q_{x+k}q_{y+k} \\
&= q_{\overline{x+k:y+k}^{1}} + \left(\frac{1}{2}-s\right) q_{x+k}q_{y+k}. \qquad (9.10.8)
\end{aligned}
$$

When the benefit is payable at the moment of death, the actuarial present value is

$$
\begin{aligned}
\bar{A}^1_{xy} &= \sum_{k=0}^{\infty} v^k\,{}_kp_{xy}\int_0^1 v^s\,{}_sp_{x+k:y+k}\,\mu(x+k+s)\,ds \\
&= \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\left[q_{\overline{x+k:y+k}^{1}}\int_0^1 (1+i)^{1-s}\,ds\right. \\
&\quad \left. + q_{x+k}\,q_{y+k}\int_0^1 (1+i)^{1-s}\left(\frac{1}{2}-s\right)ds\right] \\
&= \frac{i}{\delta}\,A^1_{xy} + \frac{1}{2}\,\frac{i}{\delta}\left(1-\frac{2}{\delta}+\frac{2}{i}\right)\sum_{k=0}^{\infty} v^{k+1}\,{}_kp_{xy}\,q_{x+k}\,q_{y+k}. \qquad (9.10.9)
\end{aligned}
$$

The second term (9.10.9) is very small relative to the total amount. It is 1/2 of the second term in (9.8.9).

9.11 Notes and References

The concept of the future-lifetime random variable, developed for a single life in previous chapters, has been extended to a survival status, particular cases of which are statuses defined by several lives. Probability distributions, actuarial present values, variances, and covariances based on these new random variables were obtained for statuses defined by two lives by adaptation of the single life theory. These concepts are developed for more than two lives in Chapter 18.

Discussions of the ideas of this chapter without the use of random variables can be found in Chapters 9–13 of Jordan (1967) and Chapters 7–8 of Neill (1977). A general analysis of laws of mortality, which simplify the formulas for actuarial functions based on more than one life, is given by Greville (1956). Exercise 9.36 illustrates Greville's analysis.

Marshall and Olkin (1967, 1988) contributed to the literature on families of bivariate distributions. In particular, they wrote about the common shock model. Frank's family of bivariate distributions is named for M. J. Frank, who developed it. This family is reviewed by Genest (1987).

Frees, Carriere, and Valdez (1996) used Frank's copula to analyze data from last-survivor annuity experience. They assumed that the marginal distributions belong

to the Gompertz family. The estimation process yielded an estimate of α of approximately -3.4. Comparing this estimate with the approximate standard error of the estimate leads to the conclusion that $T(x)$ and $T(y)$ were dependent in that experience.

The parameter α is not a conventional measure of association. The value -3.4 is associated with a positive correlation between $T(x)$ and $T(y)$. This might have been expected because in practice lives receiving payments until the last survivor dies live in the same environment.

Frees et al. also found that the assumption of independence between $T(x)$ and $T(y)$ resulted in higher estimated last-survivor annuity actuarial present values than those estimated using a model that permits dependence. The difference was in the range of 3% to 5%.

Exercises

Unless otherwise indicated, all lives in question are subject to the same table of mortality rates, and their times-until-death are independent random variables.

Section 9.2

9.1. The joint p.d.f. of $T(x)$ and $T(y)$ is given by

$$f_{T(x)T(y)}(s, t) = \frac{(n-1)(n-2)}{(1+s+t)^n} \qquad 0 < s,\ 0 < t,\ n > 2.$$

Find:

a. The joint d.f. of $T(x)$ and $T(y)$.

b. The p.d.f., d.f., and $\mu(x+s)$ for the marginal distribution of $T(x)$. Note the symmetry in s and t which implies that $T(x)$ and $T(y)$ are identically distributed.

c. The covariance and correlation coefficients of $T(x)$ and $T(x)$, given that $n > 4$.

9.2. Display the joint survival function of $[T(x), T(y)]$ where the distribution is defined in Exercise 9.1.

9.3. The future lifetime random variables $T(x)$ and $T(y)$ are independent and identically distributed with p.d.f.

$$f(t) = \frac{n-2}{(1+t)^{n-1}} \qquad n > 3,\ t > 0.$$

Determine the joint d.f. and the joint survival function.

Section 9.3

9.4. In terms of the single life probabilities ${}_np_x$ and ${}_np_y$, express
   a. The probability that (xy) will survive n years
   b. The probability that exactly one of the lives (x) and (y) will survive n years
   c. The probability that at least one of the lives (x) and (y) will survive n years
   d. The probability that (xy) will fail within n years
   e. The probability that at least one of the lives will die within n years
   f. The probability that both lives will die within n years.

9.5. Show that the probability that (x) survives n years and (y) survives $n-1$ years may be expressed either as

$$\frac{{}_np_{x:y-1}}{p_{y-1}}$$

or as

$$p_x \; {}_{n-1}p_{x+1:y}.$$

9.6. Evaluate

$$\int_0^n {}_tp_{xx} \; \mu_{xx}(t) \, dt.$$

9.7. Using the distribution of $T(x)$ and $T(y)$ shown in Exercise 9.1 display (a) the d.f., (b) the survival function, and (c) $\mathrm{E}[T(xy)]$ for $T(xy)$. Assume $n > 3$.

9.8. Use (9.3.8) to obtain the p.d.f. of $T(xy)$ for the joint distributions of $T(x)$ and $T(y)$ given in Example 9.2.3.

Section 9.4

9.9. Show

$${}_tp_{\overline{xy}} = {}_tp_{xy} + {}_tp_x \, (1 - {}_tp_y) + {}_tp_y \, (1 - {}_tp_x)$$

algebraically and by general reasoning.

9.10. Find the probability that at least one of two lives (x) and (y) will die in the year $(n+1)$. Is this the same as ${}_{n|}q_{\overline{xy}}$? Explain.

9.11. The random variables $T(x)$ and $T(y)$ have the joint p.d.f. displayed in Exercise 9.1. Find (a) the d.f. and the p.d.f. of $T(\overline{xy})$, (b) $\mathrm{E}[T(\overline{xy})]$, and (c) $\mu_{\overline{xy}}(t)$.

Section 9.5

9.12. Given that ${}_{25}p_{25:50} = 0.2$ and ${}_{15}p_{25} = 0.9$, calculate the probability that a person age 40 will survive to age 75.

9.13. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate

a. ${}_{10}p_{40:50}$
b. ${}_{10}p_{\overline{40:50}}$
c. $e_{40:50}$
d. $e_{\overline{40:50}}$
e. Var$[T(40:50)]$
f. Var$[T(\overline{40:50})]$
g. Cov$[T(40:50), T(\overline{40:50})]$
h. The correlation between $T(40:50)$ and $T(\overline{40:50})$.

9.14. Evaluate $\dfrac{d\mathring{e}_{xx}}{dx}$.

9.15. Show that the probability of two lives (30) and (40) dying in the same year can be expressed as

$$1 + e_{30:40} - p_{30}(1 + e_{31:40}) - p_{40}(1 + e_{30:41}) + p_{30:40}(1 + e_{31:41}).$$

9.16. Show that the probability of two lives (30) and (40) dying at the same age last birthday can be expressed as

$${}_{10}p_{30}(1 + e_{40:40}) - 2\,{}_{11}p_{30}\,(1 + e_{40:41}) + p_{40}\,{}_{11}p_{30}\,(1 + e_{41:41}).$$

9.17. Assume that the forces of mortality that apply to individuals I and II, respectively, are

$$\mu^{I}(x) = \log \frac{10}{9} \quad \text{for all } x$$

and

$$\mu^{II}(x) = (10 - x)^{-1} \quad \text{for } 0 \le x < 10.$$

Evaluate the probability that, if both individuals are of exact age 1, the first death will occur between exact ages 3 and 5.

Section 9.6

9.18. This is a continuation of Example 9.6.3. Exhibit (a) the d.f. and (b) the p.d.f. of $T(\overline{xy})$.

9.19. If ${}_5q_x = 0.05$ and ${}_5q_y = 0.03$, calculate the corresponding value of $F_{T(x)T(y)}(5, 5)$ using (9.6.7). Your answer will depend on the parameter α.

9.20. Use the result of Exercise 9.19 to evaluate ${}_5q_{\overline{xy}}$ for (a) $\alpha = 0$, (b) $\alpha = 3$, and (c) $\alpha = -3$. [Hint: Recall (9.4.5) and (9.3.2).]

Section 9.7

9.21. Show that

$$a_{\overline{xy}\,:\overline{n}|} = a_{\overline{n}|} + {}_{n}|a_{xy}.$$

Describe the underlying benefit.

9.22. For an actuarial present value denoted by $\bar{A}_{x:\overline{n}|}$, describe the benefit. Show that

$$\bar{A}_{x:\overline{n}|} = \bar{A}_x - \bar{A}_{x:\overline{n}|} + v^n.$$

9.23. For independent lifetimes $T(x)$ and $T(y)$, show that

$$\text{Cov}(v^{T(\overline{xy})}, v^{T(xy)}) = (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy}).$$

9.24. Express, in terms of single- and joint-life annuity values, the actuarial present value of an annuity payable continuously at a rate of 1 per year while at least one of (25) and (30) survives and is below age 50.

9.25. Express, in terms of single- and joint-life annuity values, the actuarial present value of a deferred annuity of 1 payable at the end of any year as long as either (25) or (30) is living after age 50.

9.26. Express, in terms of single- and joint-life annuity values, the actuarial present value of an n-year temporary annuity-due, payable in respect to $(\overline{xy})$, providing annual payments of 1 while both lives survive, reducing to 1/2 on the death of (x) and to 1/3 on the death of (y).

9.27. An annuity-immediate of 1 is payable to (x) as long as he lives jointly with (y) and for n years after the death of (y), except that in no event will payments be made after m years from the present time, $m > n$. Show that the actuarial present value is

$$a_{x:\overline{n}|} + {}_nE_x \, a_{x+n:y:\overline{m-n}|}.$$

9.28. Obtain an expression for the actuarial present value of a continuous annuity of 1 per annum payable while at least one of two lives (40) and (55) is living and is over age 60, but not if (40) is alive and under age 55.

9.29. A joint-and-survivor annuity to (x) and (y) is payable at an initial rate per year while (x) lives, and, if (y) survives (x), is continued at the fraction p, $1/2 \le p \le 1$, of the initial rate per year during the lifetime of (y) following the death of (x).

a. Express the actuarial present value of such an annuity-due with an initial rate of 1 per year, payable in m-thly installments, in terms of the actuarial present values of single-life and joint-life annuities.

b. A joint-and-survivor annuity to (x) and (y) and a life annuity to (x) are said to be actuarially equivalent on the basis of stated assumptions if they have equal actuarial present values on such basis. Derive an expression for the ratio of the initial payment of the joint-and-survivor annuity to the payment rate of the actuarially equivalent life annuity to (x).

9.30. Show that

a. $\bar{A}^2_{xy} = \bar{A}^1_{xy} - \delta \bar{a}_{y|x}$

b. $\dfrac{\partial}{\partial x} \bar{a}_{y|x} = \mu(x) \, \bar{a}_{y|x} - \bar{A}^2_{xy}.$

9.31. When, under Makeham's law, the status (xy) is replaced by the status (ww), show that

$$w - y = \frac{\log(c^{\Delta} + 1) - \log 2}{\log c}$$

where $\Delta = x - y \geq 0$. (This indicates that w can be obtained from the younger age y by adding an amount that is a function of $\Delta = x - y$. Such a property is referred to as a ***law of uniform seniority.***)

9.32. On the basis of your Illustrative Life Table with interest of 6% calculate $\ddot{a}_{50:60:\overline{10}|}$. In your solution, use
a. Values interpolated in the $\ddot{a}_{xx}$ table
b. Values from the $\ddot{a}_{x:x+10}$ table.

9.33. Given a mortality table that follows Makeham's law and ages x and y for which (ww) is the equivalent equal-age status, show that
a. ${}_tp_w$ is the geometric mean of ${}_tp_x$ and ${}_tp_y$
b. ${}_tp_x + {}_tp_y > 2, p_w$ for $x \neq y$
c. $a_{\overline{xy}} > a_{\overline{ww}}$ for $x \neq y$.

9.34. Given a mortality table that follows Makeham's law, show that $\bar{a}_{xy}$ is equal to the actuarial present value of an annuity with a single life (w) where $c^w = c^x + c^y$ and force of interest $\delta' = \delta + A$. Further, show that

$$\bar{A}_{xy} = \bar{A}'_w + A\bar{a}'_w$$

where the primed functions are evaluated at force of interest δ'.

9.35. Consider two mortality tables, one for males, M, and one for females, F, with

$$\mu^M(z) = 3a + \frac{3bz}{2} \quad \text{and} \quad \mu^F(z) = a + bz.$$

We wish to use a table of actuarial present values for two lives, one male and one female, each of age w, to evaluate the actuarial present value of a joint-life annuity for a male age x and a female age y. Express w in terms of x and y.

9.36. From Section 9.5 we know that if $T(x)$ and $T(y)$ are independent,

$$\bar{a}_{xy} = \int_0^{\infty} e^{-\int_0^t [\delta + \mu(x+s) + \mu(y+s)]ds}\, dt.$$

If we could find δ', k, and w such that

$$\delta + \mu(x + t) + \mu(y + t) = \delta' + k\mu(w + t) \qquad (*)$$

we would have

$$\bar{a}_{xy} = \int_0^\infty v^t({}_tp_w)^k \, dt$$

$$= \bar{a}'_{w(k)},$$

where the prime on the discount factor indicates that it is valued at force of interest δ' and $w(k)$ indicates a joint-life status with k "lives" (k is not necessarily an integer).

If $\mu(x + t) = a + b(x + t) + c(x + t)^2$, confirm that (*) is satisfied if

$$k = 2,$$

$$w = \frac{x + y}{2},$$

$$\delta' = \delta + c(x^2 + y^2 - 2w^2).$$

9.37. Find $\mathring{e}_{xy}$ if $q_x = q_y = 1$ and the deaths are uniformly distributed over the year of age for each of (x) and (y).

9.38. Let $T(x)$ and $T(y)$ be independent and uniformly distributed in the next year of age. Given that both (x) and (y) die within the next year, demonstrate that the time-of-failure of (xy) is not uniformly distributed over the year. [Hint: Show that $\Pr[T(xy) \le t | (T(x) \le 1) \cap (T(y) \le 1)] = 2t - t^2$.]

9.39. Show

$$\frac{1}{\bar{\delta}} = \frac{1}{i[1 - (i/2 - i^2/3 + i^2/4 - i^4/5 + \cdots)]}$$

$$= \frac{1}{i}\left(1 + \frac{i}{2} - \frac{i^2}{12} + \frac{i^3}{24} - \frac{19i^4}{720} + \cdots\right).$$

Hence, show

$$\frac{i}{\delta}\left(1 - \frac{2}{\delta} + \frac{2}{i}\right) \cong \frac{i}{6} - \frac{i^3}{360} + \cdots.$$

9.40. Show that if deaths are uniformly distributed over each year of age, then

$${}_{(j-1)/m}p_{xy} - {}_{j/m}p_{xy} = \frac{1}{m} q_{xy} + \frac{m + 1 - 2j}{m^2} q_x \, q_y$$

for any x and y and $j = 1, 2, 3, \ldots, m$. Hence, verify (9.8.14).

Section 9.9

9.41. Show by general reasoning that

$${}_nq^1_{xy} = {}_nq^2_{xy} + {}_nq_x \, {}_np_y.$$

When $n \to \infty$, what does the equation become?

9.42. Show that the actuarial present value for an insurance of 1 payable at the end of the year of death of (x), provided that (y) survives to the time of payment, can be expressed as $vp_y\ddot{a}_{x:y+1} - a_{xy}$.

9.43. Show that $A^1_{xy} - A^2_{xy} = A_{xy} - A_y$.

9.44. Express, in terms of actuarial present values for single life and first death contingent insurances, the net single premium for an insurance of 1 payable at the moment of death of (50), provided that (20), at that time, has died or attained age 40.

9.45. Express, in terms of actuarial present values for pure endowment and first death contingent insurances, the actuarial present value for an insurance of 1 payable at the time of the death of (x) after (y), providing (y) died during the n years preceding the death of (x).

9.46. If $\mu(x) = 1/(100 - x)$ for $0 \le x < 100$, calculate ${}_{25}q_{25:\overline{50}}^{\ 2}$.

Section 9.10

9.47. In a mortality table known to follow Makeham's law, you are given that $A = 0.003$ and $c^{10} = 3$.

a. If $\mathring{e}_{40:50} = 17$, calculate ${}_{\infty}q^1_{40:50}$.

b. Express $\bar{A}^1_{40:50}$ in terms of $\bar{A}_{40:50}$ and $\bar{a}_{40:50}$.

9.48. Given that mortality follows Gompertz's law with $\mu(x) = 10^{-4} \times 2^{x/8}$ for $x > 35$ and that by (9.10.12)

$$\bar{A}^{\ 1}_{40:48:\overline{10}|} = f\bar{A}^1_{w:\overline{10}|},$$

calculate f and w.

Miscellaneous

9.49. The survival status $(\overline{n}|)$ is one that exists for exactly n years. It has been used in conjunction with life statuses, for example, in $\bar{A}_{x:\overline{n}|}$, $\bar{A}^1_{x:\overline{n}|}$, $\bar{A}_{x:\overline{n}|}^{\ 1}$, $\ddot{a}_{x:\overline{n}|}$, $A_{xy:\overline{n}|}$. Simplify and interpret the following:

a. $\bar{a}_{x:\overline{n}|}$

b. $\bar{A}_{x:\overline{n}|}^{\ 2}$.

9.50. Use the probability rule $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ to obtain (9.4.6).

9.51. Evaluate $\dfrac{\partial}{\partial x}\,\mathring{e}_{xy}$.

9.52. The random variables $T^*(x)$, $T^*(y)$, and Z are independent and have exponential distributions with, respectively, parameters μ_1, μ_2, and λ. These three random variable are components of a common shock model.

a. Exhibit the marginal p.d.f. of $T(y)$ by evaluating

$$f_{T(y)}(t) = \int_0^t f_{T(x)T(y)}(s, t)\, ds + f_{T(x)T(y)}(t, t) + \int_t^\infty f_{T(x)T(y)}(s, t)\, ds.$$

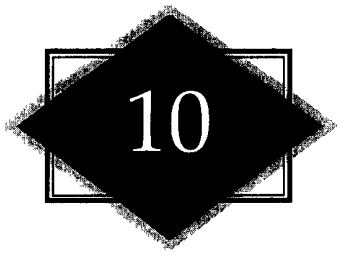
b. Exhibit the marginal survival function of $T(y)$ by evaluating

$$s_{T(y)}(t) = \int_t^\infty f_{T(y)}(u)\, du.$$

Compare the result with (9.6.4b).

c. Evaluate

$$\Pr[T(x) = T(y)] = \int_0^\infty f_{T(x)T(y)}(t, t)\, dt.$$



MULTIPLE DECREMENT MODELS

10.1 Introduction

In Chapter 9 we extended the theory of Chapters 3 through 8 from an individual life to multiple lives, subject to a single contingency of death. We now return to the case of a single life, but here subject to multiple contingencies. As an application of this extension, we observe that the number of workers for an employer is reduced when an employee withdraws, becomes disabled, dies, or retires. In manpower planning, it might be necessary to estimate only the numbers of those presently at work who will remain active to various years into the future. For this task, the model for survivorship developed in Chapter 3 is adequate, with time-until-termination of employment rather than time-until-death as the interpretation of the basic random variable. Employee benefit plans, however, provide benefits paid on termination of employment that may depend on the cause of termination. For example, the benefits on retirement often differ from those payable on death or disability. Therefore, survivorship models for employee benefit systems include random variables for both time-of-termination and cause of termination. Also, the benefit structure often depends on earnings, which is another and different kind of uncertainty that is discussed in Chapter 11.

As another application, most individual life insurances provide payment of a nonforfeiture benefit if premiums stop before the end of the specified premium payment term. A comprehensive model for such insurances incorporates both time-until-termination and cause of termination as random variables.

Disability income insurance provides periodic payments to insureds who satisfy the definition of disability contained in the policy. In some cases, the amount of the periodic payments may depend on whether the disability was caused by illness or accident. A person may cease to be an active insured by dying, withdrawing, becoming disabled, or reaching the end of the coverage period. A complete model for disability insurance incorporates a random variable for time-until-termination, when the insured ceases to be a member of the active insureds, as well as a random variable for the cause of termination.

Public health planners are interested in the analysis of mortality and survivorship by cause of death. Public health goals may be set by a study of the joint distribution of time-until-death and cause of death. Priorities in cardiovascular and cancer research were established by this type of analysis.

The main purpose of this chapter is to build a theory for studying the distribution of two random variables in regard to a single life: time-until-termination from a given status and cause of the termination. The resulting model is used in each of the applications described in this section. Within actuarial science, the termination from a given status is called ***decrement,*** and the subject of this chapter is called ***multiple decrement theory.*** Within biostatistics it is referred to as the ***theory of competing risks.***

It is also possible to develop multiple decrement theory in terms of deterministic rates and rate functions. There is some recapitulation of the theory from this point of view in Section 10.4.

10.2 Two Random Variables

Chapter 3 was devoted in part to methods for specifying and using the distribution of the continuous random variable $T(x)$, the time-until-death of (x). The same methods can be used to study time-until-termination from a status, such as employment with a particular employer, with only minor changes in vocabulary. In fact, we use the same notation $T(x)$, or T, to denote the time random variable in this new setting.

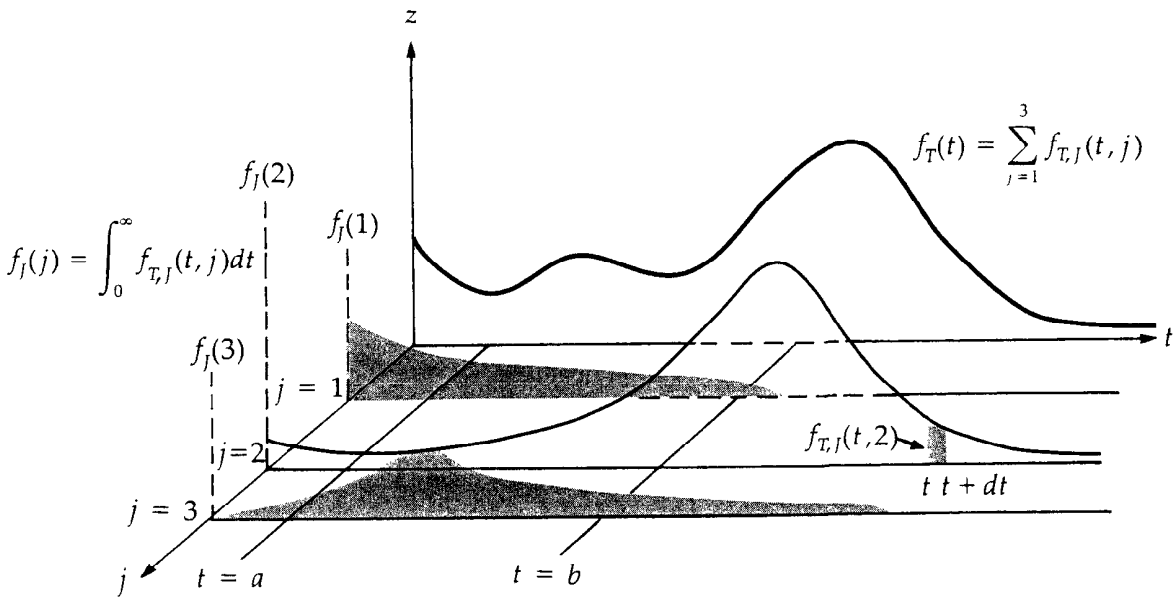
In this section we expand the basic model by introducing a second random variable, cause of decrement, to be denoted by $J(x) = J$. We assume that J is a discrete random variable.

The applications in Section 10.1 provide examples of these random variables. For employee benefit plan applications, the random variable J could be assigned the values 1, 2, 3, or 4 depending on whether termination is due to withdrawal, disability, death, or retirement, respectively. In the life insurance application, J could be assigned the values 1 or 2, depending on whether the insured dies or chooses to terminate payment of premiums. For the disability insurance application, J could be assigned the values 1, 2, 3, or 4 depending on whether the insured dies, withdraws, becomes disabled, or reaches the end of the coverage period. Finally, in the public health application, there are many possibilities for causes of decrement. For example, in a given study, J could be assigned the values 1, 2, 3, or 4 depending on whether death was caused by cardiovascular disease, cancer, accident, or all other causes.

Our purpose is to describe the joint distribution of T and J and the related marginal and conditional distributions. We denote the joint p.d.f. of T and J by $f_{T,J}(t, j)$, the marginal p.f. of J by $f_J(j)$, and the marginal p.d.f. of T by $f_T(t)$. Figure

10.2.1 illustrates these distributions. They may seem strange at first because J is a discrete random variable and T is continuous.

Graph of $f_{T,J}(t, j)$



The joint p.d.f. of T and J, $f_{T,J}(t, j)$, can be pictured as falling on m parallel sheets, as illustrated in Figure 10.2.1 for three causes of decrement ($m = 3$). There is a separate sheet for each of the m causes of decrement recognized in the model. In Figure 10.2.1 the following relations hold:

$$\sum_{j=1}^{3} f_J(j) = 1$$

and

$$\int_0^\infty f_T(t)\, dt = 1.$$

The p.d.f. $f_{T,J}(t, j)$ can be used in the usual ways to calculate the probabilities of events defined by T and J. For example,

$$f_{T,J}(t, j)\, dt = \Pr\{(t < T \leq t + dt) \cap (J = j)\} \tag{10.2.1}$$

expresses the probability of decrement due to cause j between t and $t + dt$,

$$\int_0^t f_{T,J}(s, j)\, ds = \Pr\{(0 < T \leq t) \cap (J = j)\} \tag{10.2.2}$$

expresses the probability of decrement due to cause j before time t, and

$$\sum_{j=1}^{m} \int_a^b f_{T,J}(t, j)\, dt = \Pr\{a < T \leq b\}$$

expresses the probability of decrement due to all causes between a and b.

The probability of decrement before time t due to cause j given in (10.2.2) has the special symbol

$$ {}_tq_x^{(j)} = \int_0^t f_{T,J}(s, j)\, ds \qquad t \geq 0,\ j = 1, 2, \ldots, m, \tag{10.2.3}$$

which is illustrative of the use of the superscript to denote the cause of decrement in multiple decrement theory.

The use of information given at age x to select a distribution is similar to the concepts in Chapter 3. If being in the survival status at age x is the only information available, then an aggregate distribution would be used. On the other hand, if there is additional information, then the distribution would be a select distribution and the age of selection would be enclosed in brackets.

By the definition of the marginal distribution for J, appearing as $f_J(j)$ in the (j, z) plane of Figure 10.2.1, we have the probability of decrement due to cause j at any time in the future to be

$$f_J(j) = \int_0^\infty f_{T,J}(s, j)\, ds = {}_\infty q_x^{(j)} \qquad j = 1, 2, \ldots, m. \tag{10.2.4}$$

This is new and without a counterpart in Chapter 3, unlike the marginal p.d.f. for T, $f_T(t)$ in the (t, z) plane of Figure 10.2.1. For $f_T(t)$, and the d.f., $F_T(t)$, we have for $t \geq 0$

$$f_T(t) = \sum_{j=1}^{m} f_{T,J}(t, j) \tag{10.2.5}$$

and

$$F_T(t) = \int_0^t f_T(s)\, ds.$$

The notations introduced in Chapter 3 for the functions of the distribution of the future-lifetime random variable, T, can be extended to accommodate those of the time-until-decrement random variable of the multiple decrement model. Using the superscript (τ) to indicate that a function refers to all causes, or total force, of decrement, we obtain

$$ {}_tq_x^{(\tau)} = \Pr\{T \leq t\} = F_T(t) = \int_0^t f_T(s)\, ds, \tag{10.2.6}$$

$$ {}_tp_x^{(\tau)} = \Pr\{T > t\} = 1 - {}_tq_x^{(\tau)}, \tag{10.2.7}$$

$$\mu_x^{(\tau)}(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x^{(\tau)}$$

$$= -\frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tp_x^{(\tau)}$$

$$= -\frac{d}{dt} \log {}_tp_x^{(\tau)}, \tag{10.2.8}$$

and

$${}_tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(s)\, ds}. \tag{10.2.9}$$

Mathematically, these functions for the random variable T of this chapter are identical to those for the T of Chapter 3; the difference is in their interpretation in the applications. The choice of the symbol $\mu_x^{(\tau)}(t)$ for the force of total decrement is influenced by these applications. For example, in pension applications (x) is an age of entry into the pension plan, and although no special selection information may be available, subsequent causes of decrement may depend on this age.

As with the applications in previous chapters, the statement in (10.2.1) can be analyzed by conditioning on survival in the given status to time t. In this way, we have

$$f_{T,J}(t, j)\, dt = \Pr\{T > t\} \Pr\{[(t < T \leq t + dt) \cap (J = j)] | T > t\}. \tag{10.2.10}$$

By analogy with (3.2.12) this suggests the definition of the ***force of decrement due to cause j*** as

$$\mu_x^{(j)}(t) = \frac{f_{T,J}(t, j)}{1 - F_T(t)} = \frac{f_{T,J}(t, j)}{{}_tp_x^{(\tau)}}. \tag{10.2.11}$$

The force of decrement at age $x + t$ due to cause j has a conditional probability interpretation. It is the value of the joint conditional p.d.f. of T and J at $x + t$ and j, given survival to $x + t$. Then (10.2.10) can be rewritten as

$$f_{T,J}(t, j)\, dt = {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \qquad j = 1, 2, \ldots, m,\ t \geq 0. \qquad \text{(10.2.10) restated}$$

Restated in words,

(the probability of decrement between t and $t + dt$ due to cause j) = (the probability, ${}_tp_x^{(\tau)}$, that (x) remains in the given status until time t) × (the conditional probability, $\mu_x^{(j)}(t)$ that decrement occurs between t and $t + dt$ due to cause j, given that decrement has not occurred before time t).

It follows, from differentiation of (10.2.3) and use of (10.2.11), that

$$\mu_x^{(j)}(t) = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x^{(j)}. \tag{10.2.12}$$

Now, from (10.2.6), (10.2.5), and (10.2.3),

$$
\begin{aligned}
{}_tq_x^{(\tau)} &= \int_0^t f_T(s)\, ds = \int_0^t \sum_{j=1}^m f_{T,J}(s, j)\, ds \\
&= \sum_{j=1}^m \int_0^t f_{T,J}(s, j)\, ds = \sum_{j=1}^m {}_tq_x^{(j)}.
\end{aligned} \tag{10.2.13}
$$

That the first and last members of (10.2.13) are equal is immediately interpretable. Combining (10.2.8), (10.2.13), and (10.2.12), we have

$$
\mu_x^{(\tau)}(t) = \sum_{j=1}^m \mu_x^{(j)}(t); \tag{10.2.14}
$$

that is, the total force of decrement is the sum of the forces of decrement due to the m causes.

We can summarize the definitions here by expressing the joint, marginal, and conditional p.d.f.'s in actuarial notation and repeating the defining equation numbers:

$$
f_{T,J}(t, j) = {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t), \qquad \text{(10.2.11) restated}
$$

$$
f_J(j) = {}_\infty q_x^{(j)}, \qquad \text{(10.2.4) restated}
$$

$$
f_T(t) = {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t). \qquad \text{(10.2.8) restated}
$$

The conditional p.f. of J, given decrement at time t, is

$$
\begin{aligned}
f_{J|T}(j|t) &= \frac{f_{T,J}(t, j)}{f_T(t)} = \frac{{}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)}{{}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)} \\
&= \frac{\mu_x^{(j)}(t)}{\mu_x^{(\tau)}(t)}.
\end{aligned} \tag{10.2.15}
$$

Finally, we note that the probability in (10.2.3) can be rewritten as

$$
{}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)}\, \mu_x^{(j)}(s)\, ds. \tag{10.2.16}
$$

Example 10.2.1

Consider a multiple decrement model with two causes of decrement; the forces of decrement are given by

$$
\mu_x^{(1)}(t) = \frac{t}{100} \qquad t \geq 0,
$$

$$
\mu_x^{(2)}(t) = \frac{1}{100} \qquad t \geq 0.
$$

For this model, calculate the p.f. (or p.d.f.) for the joint, marginal, and conditional distributions.

Solution:

Since

$$\mu_x^{(\tau)}(s) = \mu_x^{(1)}(s) + \mu_x^{(2)}(s) = \frac{s+1}{100},$$

the survival probability ${}_tp_x^{(\tau)}$ is

$$\begin{aligned} {}_tp_x^{(\tau)} &= \exp\left(-\int_0^t \frac{s+1}{100}\,ds\right) \\ &= \exp\left(\frac{-(t^2+2t)}{200}\right) \qquad t \geq 0, \end{aligned}$$

and the joint p.d.f. of T and J is

$$f_{T,J}(t, j) = \begin{cases} \dfrac{t}{100}\exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \geq 0,\ j = 1 \\ \dfrac{1}{100}\exp\left[\dfrac{-(t^2+2t)}{200}\right] & t \geq 0,\ j = 2. \end{cases}$$

The marginal p.d.f. of T is

$$f_T(t) = \sum_{j=1}^{2} f_{T,J}(t, j) = \frac{t+1}{100}\exp\left[\frac{-(t^2+2t)}{200}\right] \qquad t \geq 0,$$

and the marginal p.f. of J is

$$f_J(j) = \begin{cases} \displaystyle\int_0^\infty f_{T,J}(t, 1)\,dt & j = 1 \\ \displaystyle\int_0^\infty f_{T,J}(t, 2)\,dt & j = 2. \end{cases}$$

It is somewhat easier to evaluate $f_J(2)$. In the following development, $\Phi(x)$ is the d.f. for the standard normal distribution $N(0, 1)$. By completing the square we have

$$\begin{aligned} f_J(2) &= \frac{1}{100}\,e^{0.005}\int_0^\infty \exp\left[\frac{-(t+1)^2}{200}\right]dt \\ &= \frac{1}{100}\,e^{0.005}\sqrt{2\pi}\,10\int_0^\infty \frac{1}{\sqrt{2\pi}\,10}\exp\left[\frac{-(t+1)^2}{200}\right]dt. \end{aligned}$$

We now make the change of variable $z = (t+1)/10$ and obtain

$$\begin{aligned} f_J(2) &= \frac{1}{10}\,e^{0.005}\sqrt{2\pi}\int_{0.1}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-z^2}{2}\right)dz \\ &= \frac{1}{10}\,e^{0.005}\sqrt{2\pi}\,[1 - \Phi(0.1)] \\ &= 0.1159. \end{aligned}$$

Therefore $f_J(1) = 0.8841$. Finally, the conditional p.f. of J, given decrement at t, is derived from (10.2.15) as

$$f_{J|T}(1|t) = \frac{t}{t+1}$$

and

$$f_{J|T}(2|t) = \frac{1}{t+1}.$$

▼

Example 10.2.2

For the joint distribution of T and J specified in Example 10.2.1, calculate $\mathrm{E}[T]$ and $\mathrm{E}[T|J = 2]$.

Solution:

Using the marginal p.d.f. $f_T(t)$, we have

$$\mathrm{E}[T] = \int_0^\infty t \left\{ \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200} \right] \right\} dt.$$

Integration by parts, as in (3.5.1), yields

$$\begin{aligned} \mathrm{E}[T] &= -t \exp\left[\frac{-(t^2+2t)}{200} \right] \Bigg|_0^\infty \\ &\quad + \int_0^\infty \exp\left[\frac{-(t^2+2t)}{200} \right] dt \\ &= 0 + 100 f_J(2), \end{aligned}$$

hence

$$\mathrm{E}[T] = 11.59.$$

Using the conditional p.d.f. $f_{T,J}(t, 2)/f_J(2)$, we have

$$\mathrm{E}[T|J = 2] = \int_0^\infty t \left\{ 100^{-1} \exp\left[\frac{-(t^2+2t)}{200} \right] \right\} (0.1159)^{-1} \, dt.$$

This integral may be evaluated as follows:

$$\begin{aligned} \mathrm{E}[T|J = 2] &= \mathrm{E}[(T+1) - 1 | J = 2] \\ &= (0.1159)^{-1} \int_0^\infty \frac{t+1}{100} \exp\left[\frac{-(t^2+2t)}{200} \right] dt - 1 \\ &= -(0.1159)^{-1} \exp\left[\frac{-(t^2+2t)}{200} \right] \Bigg|_0^\infty - 1 \\ &= 7.63. \end{aligned}$$

The point of Examples 10.2.1 and 10.2.2 is that once the joint distribution of T and J is specified, marginal and conditional distributions can be derived, and the moments of these distributions determined. ▼

In some instances, a particular application may require a modification of the above model. A continuous distribution for time-until-termination, T, is inadequate

in applications where there is a time at which there is a positive probability of decrement. One example of this is a pension plan with a mandatory retirement age, an age at which all remaining active employees must retire. A second example is term life insurance in which there is typically no benefit paid on withdrawal. Thus, after a premium is paid, none of the remaining insureds withdraw until the next premium due date. Here we do not attempt to extend the notation to cover such situations. However, in Section 10.7 we describe extended models for each of these examples.

The random variable K, the curtate-future-years before decrement of (x), is defined as in Chapter 3 to be the greatest integer less than T. Using (10.2.1) and (10.2.11), we can write the joint p.f. K and J as

$$\begin{aligned}\Pr\{(K = k) \cap (J = j)\} &= \Pr\{(k < T \leq k + 1) \cap (J = j)\} \\ &= \int_k^{k+1} {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt. \end{aligned} \tag{10.2.17}$$

Rewriting ${}_tp_x^{(\tau)}$ of the integrand in the exponential form of (10.2.9) and factoring it into two factors changes (10.2.17) to

$$= {}_kp_x^{(\tau)} \int_k^{k+1} e^{-\int_k^t \mu_x^{(\tau)}(u)\, du}\, \mu_x^{(j)}(t)\, dt.$$

Changing the variables of the integrations by $r = u - k$ and $s = t - k$ yields

$$= {}_kp_x^{(\tau)} \int_0^1 e^{-\int_0^s \mu_x^{(\tau)}(k+r)\, dr}\, \mu_x^{(j)}(k + s)\, ds.$$

Thus far we have done manipulations that hold in all tables. If we are using an aggregate or ultimate (a nonselect) table where the forces of decrement depend on an initial age and the duration only through their sum, that is, the ***attained age***, then for τ and all j,

$$\mu_x(k + s) = \mu_{x+k}(s) \qquad \text{for all } x,\ k, \text{ and } s \geq 0,$$

and (10.2.17) may be written

$${}_kp_x^{(\tau)} \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(j)}(s)\, ds = {}_kp_x^{(\tau)}\, q_{x+k}^{(j)}. \tag{10.2.18}$$

The probability of decrement from all causes between ages $x + k$ and $x + k + 1$, given survival to age $x + k$, is denoted by $q_{x+k}^{(\tau)}$, and it follows that

$$\begin{aligned} q_{x+k}^{(\tau)} &= \int_0^1 {}_sp_{x+k}^{(\tau)}\, \mu_{x+k}^{(\tau)}(s)\, ds \\ &= \int_0^1 {}_sp_{x+k}^{(\tau)} \sum_{j=1}^{m} \mu_{x+k}^{(j)}(s)\, ds \\ &= \sum_{j=1}^{m} q_{x+k}^{(j)}. \end{aligned} \tag{10.2.19}$$

An examination of (10.2.18) and (10.2.19) discloses why multiple decrement theory is also called the theory of competing risks. The probability of decrement between ages $x + k$ and $x + k + 1$ due to cause j depends on ${}_sp_{x+k}^{(\tau)}$, $0 \le s \le 1$, and thus on all the component forces. When the forces for other decrements are increased, ${}_sp_{x+k}^{(\tau)}$ is reduced, and then $q_{x+k}^{(j)}$ is also decreased.

10.3 Random Survivorship Group

Let us consider a group of $l_a^{(\tau)}$ lives age a years. Each life is assumed to have a distribution of time-until-decrement and cause of decrement specified by the p.d.f.

$$f_{T,J}(t, j) = {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t) \qquad t \ge 0,\ j = 1, 2, \ldots, m.$$

We denote by ${}_n\mathfrak{D}_x^{(j)}$ the random variable equal to the number of lives who leave the group between ages x and $x + n$, $x \ge a$, from cause j. We denote $\mathrm{E}[{}_n\mathfrak{D}_x^{(j)}]$ by ${}_nd_x^{(j)}$ and obtain

$$\begin{aligned} {}_nd_x^{(j)} &= \mathrm{E}[{}_n\mathfrak{D}_x^{(j)}] \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt. \end{aligned} \tag{10.3.1a}$$

As usual, if $n = 1$, we delete the prefixes on ${}_n\mathfrak{D}_x^{(j)}$ and ${}_nd_x^{(j)}$. We note that

$${}_n\mathfrak{D}_x^{(\tau)} = \sum_{j=1}^{m} {}_n\mathfrak{D}_x^{(j)}$$

and define

$${}_nd_x^{(\tau)} = \mathrm{E}[{}_n\mathfrak{D}_x^{(\tau)}] = \sum_{j=1}^{m} {}_nd_x^{(j)}. \tag{10.3.1b}$$

Then, using (10.3.1a), we have

$$\begin{aligned} {}_nd_x^{(\tau)} &= l_a^{(\tau)} \sum_{j=1}^{m} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(j)}(t)\, dt \\ &= l_a^{(\tau)} \int_{x-a}^{x+n-a} {}_tp_a^{(\tau)}\, \mu_a^{(\tau)}(t)\, dt. \end{aligned} \tag{10.3.2}$$

If $\mathfrak{L}^{(\tau)}(x)$ is defined as the random variable equal to the number of survivors at age x out of the $l_a^{(\tau)}$ lives in the original group at age a, then by analogy with (3.3.1) we can write

$$\begin{aligned} l_x^{(\tau)} &= \mathrm{E}[\mathfrak{L}^{(\tau)}(x)] \\ &= l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}. \end{aligned} \tag{10.3.3}$$

We recognize the integral of (10.3.1a) with $n = 1$ as the integral of (10.2.17) with $x = a$ and $k = x - a$. Thus for a nonselect table, we have from (10.2.18)

$$d_x^{(j)} = l_a^{(\tau)}\, {}_{x-a}p_a^{(\tau)}\ q_x^{(j)} = l_x^{(\tau)}\, q_x^{(j)}. \tag{10.3.4}$$

This result lets us display a table of $p_x^{(\tau)}$ and $q_x^{(j)}$ values in a corresponding table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values. Either table is called a ***multiple decrement table.***

Example 10.3.1

Construct a table of $l_x^{(\tau)}$ and $d_x^{(j)}$ values corresponding to the probabilities of decrement given below.

x	$q_x^{(1)}$	$q_x^{(2)}$
65	0.02	0.05
66	0.03	0.06
67	0.04	0.07
68	0.05	0.08
69	0.06	0.09
70	0.00	1.00

Although this display is designed for computational ease, it may be roughly suggestive of a double decrement situation with cause 1 related to death and cause 2 to retirement. It appears that, in this case, 70 is the mandatory retirement age.

Solution:

We assume the arbitrary value of $l_{65}^{(\tau)} = 1{,}000$ and use (10.3.4) as indicated below.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(\tau)}$	$p_x^{(\tau)}$	$l_x^{(\tau)} = l_{x-1}^{(\tau)} p_{x-1}^{(\tau)}$	$d_x^{(1)} = l_x^{(\tau)} q_x^{(1)}$	$d_x^{(2)} = l_x^{(\tau)} q_x^{(2)}$
65	0.02	0.05	0.07	0.93	1 000.00	20.00	50.00
66	0.03	0.06	0.09	0.91	930.00	27.90	55.80
67	0.04	0.07	0.11	0.89	846.30	33.85	59.24
68	0.05	0.08	0.13	0.87	753.21	37.66	60.26
69	0.06	0.09	0.15	0.85	655.29	39.32	58.98
70	0.00	1.00	1.00	0.00	557.00	0.00	557.00

Note, as a check on the calculations, that $l_{x+1}^{(\tau)} = l_x^{(\tau)} - d_x^{(1)} - d_x^{(2)}$, except for rounding error.

We continue this example with the evaluation, from first principles, of several probabilities:

$$ {}_2p_{65}^{(\tau)} = p_{65}^{(\tau)} p_{66}^{(\tau)} = (0.93)(0.91) = 0.8463, $$

$$ {}_{2|}q_{66}^{(1)} = p_{66}^{(\tau)} p_{67}^{(\tau)} q_{68}^{(1)} = (0.91)(0.89)(0.05) = 0.0405, $$

$$ {}_2q_{67}^{(2)} = q_{67}^{(2)} + p_{67}^{(\tau)} q_{68}^{(2)} = 0.07 + (0.89)(0.08) = 0.1412. $$

The last three columns of the above table may be used to obtain the same probabilities. The answers agree to four decimal places:

$$ {}_2p_{65}^{(\tau)} = \frac{l_{67}^{(\tau)}}{l_{65}^{(\tau)}} = \frac{846.30}{1{,}000.00} = 0.8463, $$

$$ {}_{2|}q_{66}^{(1)} = \frac{d_{68}^{(1)}}{l_{66}^{(\tau)}} = \frac{37.66}{930.00} = 0.0405, $$

$$ {}_2q_{67}^{(2)} = \frac{d_{67}^{(2)} + d_{68}^{(2)}}{l_{67}^{(\tau)}} = \frac{59.24 + 60.26}{846.30} = 0.1412. $$ ▼

10.4 Deterministic Survivorship Group

The total force of decrement can also be viewed as a total (nominal annual) rate of decrement rather than as a conditional probability density. In this view, where we assume a continuous model, a group of $l_a^{(\tau)}$ lives advance through age subject to deterministic forces of decrement $\mu_a^{(\tau)}(y - a)$, $y \geq a$. The number of survivors to age x from the original group of $l_a^{(\tau)}$ lives at age a is given by

$$ l_x^{(\tau)} := l_a^{(\tau)} \exp\left[-\int_a^x \mu_a^{(\tau)}(y - a)\, dy\right], \tag{10.4.1} $$

and the total decrement between ages x and $x + 1$ is

$$ \begin{aligned} d_x^{(\tau)} &= l_x^{(\tau)} - l_{x+1}^{(\tau)} \\ &= l_x^{(\tau)}\left(1 - \frac{l_{x+1}^{(\tau)}}{l_x^{(\tau)}}\right) \\ &= l_x^{(\tau)}\left\{1 - \exp\left[-\int_x^{x+1} \mu_a^{(\tau)}(y - a)\, dy\right]\right\} \\ &= l_x^{(\tau)}(1 - p_x^{(\tau)}) \\ &= l_x^{(\tau)} q_x^{(\tau)}. \end{aligned} \tag{10.4.2} $$

Further, by definition or from differentiating (10.4.1), we have

$$ \mu_a^{(\tau)}(x - a) = -\frac{1}{l_x^{(\tau)}} \frac{dl_x^{(\tau)}}{dx}. \tag{10.4.3} $$

These formulas are analogous to those for life tables in Section 3.4. Here $q_x^{(\tau)}$ is the effective annual total rate of decrement for the year of age x to $x + 1$ equivalent to the forces $\mu_a^{(\tau)}(y - a)$, $x \leq y < x + 1$.

Consider next m causes of decrement and assume that the $l_x^{(\tau)}$ survivors to age x will, at future ages, be fully depleted by these m forms of decrement. Then the $l_x^{(\tau)}$ survivors can be visualized as falling into distinct subgroups $l_x^{(j)}$, $j = 1, 2, \ldots, m$, where $l_x^{(j)}$ denotes the number from the $l_x^{(\tau)}$ survivors who will terminate at future ages due to cause j, so that

$$ l_x^{(\tau)} = \sum_{j=1}^{m} l_x^{(j)}. \tag{10.4.4} $$

We define the force of decrement at age x due to cause j by

$$\mu_a^{(j)}(x - a) = \lim_{h \to 0} \frac{l_x^{(j)} - l_{x+h}^{(j)}}{h l_x^{(\tau)}}$$

where $l_x^{(\tau)}$, not $l_x^{(j)}$, appears in the denominator. This yields

$$\mu_a^{(j)}(x - a) = -\frac{1}{l_x^{(\tau)}} \frac{dl_x^{(j)}}{dx}. \tag{10.4.5}$$

From (10.4.3)–(10.4.5) it follows that

$$\mu_a^{(\tau)}(x - a) = -\frac{1}{l_x^{(\tau)}} \frac{d}{dx} \sum_{j=1}^{m} l_x^{(j)} = \sum_{j=1}^{m} \mu_a^{(j)}(x - a). \tag{10.4.6}$$

Formula (10.4.5), substituting y for x, can be written as

$$-dl_y^{(j)} = l_y^{(\tau)} \mu_a^{(j)}(y - a)\, dy,$$

and integration from $y = x$ to $y = x + 1$ gives

$$l_x^{(j)} - l_{x+1}^{(j)} = d_x^{(j)} = \int_x^{x+1} l_y^{(\tau)} \mu_a^{(j)}(y - a)\, dy. \tag{10.4.7}$$

Summation over $j = 1, 2, \ldots, m$ yields

$$l_x^{(\tau)} - l_{x+1}^{(\tau)} = d_x^{(\tau)} = \int_x^{x+1} l_y^{(\tau)} \mu_a^{(\tau)}(y - a)\, dy. \tag{10.4.8}$$

Further, from division of formula (10.4.7) by $l_x^{(\tau)}$, we have

$$\frac{d_x^{(j)}}{l_x^{(\tau)}} = \int_x^{x+1} {}_{y-x}p_x^{(\tau)} \mu_a^{(j)}(y - a)\, dy = q_x^{(j)}. \tag{10.4.9}$$

Here $q_x^{(j)}$ is defined as the proportion of the $l_x^{(\tau)}$ survivors to age x who terminate due to cause j before age $x + 1$ when all m causes of decrement are operating.

As was the case for life tables, the deterministic model provides an alternative language and conceptual framework for multiple decrement theory.

10.5 Associated Single Decrement Tables

For each of the causes of decrement recognized in a multiple decrement model, it is possible to define a single decrement model that depends only on the particular cause of decrement. We define the ***associated single decrement model*** functions as follows:

$$\begin{aligned} {}_tp_x'^{(j)} &= \exp\left[-\int_0^t \mu_x^{(j)}(s)\, ds\right], \\ {}_tq_x'^{(j)} &= 1 - {}_tp_x'^{(j)}. \end{aligned} \tag{10.5.1}$$

Quantities such as ${}_tq_x'^{(j)}$ are called ***net probabilities of decrement*** in biostatistics because they are net of other causes of decrement. However, many other names have been given to the same quantity. One is ***independent rate of decrement,*** chosen

because cause j does not compete with other causes in determining ${}_tq_x'^{(j)}$. The term we use for ${}_tq_x'^{(j)}$ is ***absolute rate of decrement.*** The use of the word ***rate*** in describing ${}_tq_x'^{(j)}$ stems from a desire to distinguish between q and q'. The symbol ${}_tq_x^{(j)}$ denotes a probability of decrement for cause j between ages x and $x + t$, and we will show that it differs from ${}_tq_x'^{(j)}$. In addition, ${}_tp_x'^{(j)}$, unlike ${}_tp_x^{(\tau)}$, is not necessarily a survivorship function, because it is not required that $\lim_{t\to\infty} {}_tp_x'^{(j)} = 0$.

While

$$\int_0^\infty \mu_x^{(\tau)}(t)\, dt = \infty,$$

we can conclude from (10.2.14) only that

$$\int_0^\infty \mu_x^{(j)}(t)\, dt = \infty$$

for at least one j. There may be causes of decrement for which this integral is finite.

We seldom have an opportunity to observe the operation of a random survival system in which a single cause of decrement operates. In an employee benefit plan, retirement, disabilities, and voluntary terminations make it impossible to directly observe the operation of a single decrement model for mortality during active service. In biostatistical applications random withdrawals from observation and arbitrary ending of the period of study may prevent the observation of mortality alone operating on a group of lives.

As we see in Section 10.6, a usual first step in constructing a multiple decrement model is to select absolute rates of decrement and to make assumptions concerning the incidence of the decrements within any single year of age to obtain probabilities $q_x^{(j)}$. The converse problem of obtaining absolute rates from the probabilities also involves assumptions about the incidence of the decrements. These assumptions are implicit in statistical methods for estimating absolute rates and will be discussed in Section 10.5.5.

In the next subsection we examine a number of relationships between a multiple decrement table and its associated single decrement tables. Then we examine a number of special assumptions about incidence of decrement over the year of age and note some implied relationships. In Section 10.5.5 some of the statistical issues in estimating a multiple decrement distribution are examined.

10.5.1 Basic Relationships

First, note that since

$${}_tp_x^{(\tau)} = \exp\left\{-\int_0^t [\mu_x^{(1)}(s) + \mu_x^{(2)}(s) + \cdots + \mu_x^{(m)}(s)]\, ds\right\},$$

we have

$$ {}_tp_x^{(\tau)} = \prod_{i=1}^{m} {}_tp_x^{\prime(i)}. \tag{10.5.2} $$

This result does not involve any approximation. It is based on the definition of an associated single decrement table where the sole force of decrement is equal to the force for that decrement in the multiple decrement model. We require that it hold for any method used to construct a multiple decrement table from a set of absolute rates of decrement.

Now compare the size of the absolute rates and the probabilities. From (10.5.2) we see, if some cause other than j is operating, that

$$ {}_tp_x^{\prime(j)} \geq {}_tp_x^{(\tau)}. $$

This implies

$$ {}_tp_x^{\prime(j)}\, \mu_x^{(j)}(t) \geq {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t), $$

and if these functions are integrated with respect to t over the interval (0, 1), we obtain

$$ q_x^{\prime(j)} = \int_0^1 {}_tp_x^{\prime(j)}\, \mu_x^{(j)}(t)\, dt \geq \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt = q_x^{(j)}. \tag{10.5.3} $$

The magnitude of other forces of decrement can cause ${}_tp_x^{\prime(j)}$ to be considerably greater than ${}_tp_x^{(\tau)}$, and thus there can be corresponding differences between the absolute rates and the probabilities.

10.5.2 Central Rates of Multiple Decrement

There is one function of the multiple decrement model that is quite close to the corresponding function for an associated single decrement model. To introduce this function, we return to a mortality table and recall the central rate of mortality, or central-death-rate at age x, denoted by m_x and defined in (3.5.13) by

$$ m_x = \frac{\int_0^1 {}_tp_x\, \mu_x(t)\, dt}{\int_0^1 {}_tp_x\, dt} = \frac{\int_0^1 l_{x+t}\, \mu_x(t)\, dt}{\int_0^1 l_{x+t}\, dt} = \frac{d_x}{L_x}. \tag{10.5.4} $$

Thus, m_x is a weighted average of the force of mortality between ages x and $x + 1$, and this justifies the term ***central rate.***

Such central rates can be defined in a multiple decrement context. The ***central rate of decrement from all causes*** is defined by

$$ m_x^{(\tau)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)\, dt}{\int_0^1 {}_tp_x^{(\tau)}\, dt} \tag{10.5.5} $$

and is a weighted average of $\mu_x^{(\tau)}(t)$, $0 \le t < 1$. Similarly, the ***central rate of decrement from cause j*** is

$$m_x^{(j)} = \frac{\int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x^{(\tau)}\, dt} \tag{10.5.6}$$

and is a weighted average of $\mu_x^{(j)}(t)$, $0 \le t < 1$. Clearly,

$$m_x^{(\tau)} = \sum_{j=1}^{m} m_x^{(j)}.$$

The corresponding central rate for the associated single decrement table is given by

$$m_x'^{(j)} = \frac{\int_0^1 {}_tp_x'^{(j)}\, \mu_x^{(j)}(t)\, dt}{\int_0^1 {}_tp_x'^{(j)}\, dt}. \tag{10.5.7}$$

This is again a weighted average of $\mu_x^{(j)}(t)$ over the same age range, the weight function now ${}_tp_x'^{(j)}$ rather than ${}_tp_x^{(\tau)}$. If the force $\mu_x^{(j)}(t)$ is constant for $0 \le t < 1$, we have $m_x^{(j)} = m_x'^{(j)} = \mu_x^{(j)}(0)$. If $\mu_x^{(j)}(t)$ is an increasing function of t, then ${}_tp_x'^{(j)}$ gives more weight to higher values than does ${}_tp_x^{(\tau)}$, and $m_x'^{(j)} > m_x^{(j)}$. If $\mu_x^{(j)}(t)$ is a decreasing function of t, then $m_x'^{(j)} < m_x^{(j)}$. See Exercise 10.33 for a more formal treatment of these statements.

Central rates provide a convenient but approximate means of proceeding from the $q_x'^{(j)}$ to the $q_x^{(j)}$, $j = 1, 2, \ldots, m$, and vice versa. This is illustrated in Exercise 10.18.

10.5.3 Constant Force Assumption for Multiple Decrements

Let us examine specific assumptions concerning the incidence of decrements. First, let us use an assumption of a constant force for decrement j and for the total decrement over the interval $(x, x + 1)$. This implies

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$$

and

$$\mu_x^{(\tau)}(t) = \mu_x^{(\tau)}(0) \qquad 0 \le t < 1.$$

Then, for $0 \le s \le 1$, we have

$$\begin{aligned} {}_sq_x^{(j)} &= \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(j)}(t)\, dt \\ &= \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)} \int_0^s {}_tp_x^{(\tau)}\, \mu_x^{(\tau)}(t)\, dt = \frac{\mu_x^{(j)}(0)}{\mu_x^{(\tau)}(0)}\, {}_sq_x^{(\tau)}. \end{aligned} \tag{10.5.8}$$

But also for any r in $(0, 1)$, under the constant force assumption,

$$r\mu_x^{(\tau)}(0) = -\log {}_rp_x^{(\tau)}$$

and

$$r\mu_x^{(j)}(0) = -\log {}_rp_x'^{(j)},$$

so that from (10.5.8),

$${}_sq_x^{(j)} = \frac{\log {}_rp_x'^{(j)}}{\log {}_rp_x^{(\tau)}} \, {}_sq_x^{(\tau)}. \tag{10.5.9}$$

Equation (10.5.9) can be rearranged as

$${}_rp_x'^{(j)} = ({}_rp_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}}$$

and then in the limit as r goes to 1 can be solved for $q_x'^{(j)}$ to give

$$q_x'^{(j)} = 1 - (p_x^{(\tau)})^{{}_sq_x^{(j)}/{}_sq_x^{(\tau)}}. \tag{10.5.10}$$

If the constant force assumption holds for all decrements (and then automatically for the total decrement), (10.5.9), as r and s approach 1, together with (10.5.2), can be used for calculating $q_x^{(j)}$ from given values of $q_x'^{(j)}$, $j = 1, 2, \ldots, m$. Also, (10.5.10) is useful for obtaining absolute rates from a set of probabilities of decrement. Note that for (10.5.9) and (10.5.10) special treatment is required if $p_x'^{(j)}$ or $p_x^{(\tau)}$ equals 0.

10.5.4 Uniform Distribution Assumption for Multiple Decrements

Formula (10.5.10) holds under alternative assumptions. One of these is that both decrement j and total decrement, in the multiple decrement context, have a uniform distribution of decrement over the interval $(x, x + 1)$. Thus we assume that

$${}_tq_x^{(j)} = t \, q_x^{(j)}$$

and

$${}_tq_x^{(\tau)} = t \, q_x^{(\tau)}.$$

Also under the given assumption, we see from (10.2.12) that

$${}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) = q_x^{(j)} \tag{10.5.11}$$

and

$$\mu_x^{(j)}(t) = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - t \, q_x^{(\tau)}}.$$

Then

$$\begin{aligned} {}_sp_x'^{(j)} &= \exp\left[-\int_0^s \mu_x^{(j)}(t) \, dt\right] \\ &= \exp\left(-\int_0^s \frac{q_x^{(j)}}{1 - t \, q_x^{(\tau)}} \, dt\right) \\ &= \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \log (1 - s q_x^{(\tau)})\right] \\ &= ({}_sp_x^{(\tau)})^{q_x^{(j)}/q_x^{(\tau)}}. \end{aligned} \tag{10.5.12}$$

At $s = 1$, (10.5.10) and (10.5.12) yield the same equation relating $q_x'^{(j)}$ with $q_x^{(j)}$ and $q_x^{(\tau)}$. As a result, (10.5.9) with $r = 1$ can be used to obtain $q_x^{(j)}$. Exercise 10.22 provides additional insights into the connection between the developments in Sections 10.5.3 and 10.5.4.

Example 10.5.1

Continue Example 10.3.1 evaluating $q_x'^{(1)}$ and $q_x'^{(2)}$ by (10.5.10).

Solution:

By (10.5.10), the following results are obtained.

x	$q_x^{(1)}$	$q_x^{(2)}$	$q_x'^{(1)}$	$q_x'^{(2)}$
65	0.02	0.05	0.02052	0.05052
66	0.03	0.06	0.03095	0.06094
67	0.04	0.07	0.04149	0.07147
68	0.05	0.08	0.05215	0.08213
69	0.06	0.09	0.06294	0.09291
70	0.00	1.00	—	—

At age 70, the rates depend on mandatory retirement, and there is no particular need for $q_{70}'^{(1)}$, $q_{70}'^{(2)}$ although they could be identified, respectively, using $q_{70}^{(1)}$ and $q_{70}^{(2)}$. ▼

10.5.5 Estimation Issues

The definition of the absolute rate of decrement given in (10.5.1) depends on the force of decrement in multiple decrement theory as defined in (10.2.11). From definition (10.5.1), the developments in this section and their application in constructing multiple decrement distributions from assumed absolute rates of decrement follow. Questions remain, however, about the interpretation and estimation of ${}_tp_x'^{(j)}$.

If the joint p.d.f. $f_{T,J}(t, j)$ is known, then the survival function and forces of decrement are determined using the formulas of Section 10.2. For example, (10.2.13) follows as a consequence of the assumption that decrements occur from m mutually exclusive causes. The issue is, under what conditions can information obtained in a single decrement environment be used to construct the distribution of (T, J)?

We will illustrate this issue by considering two causes of decrement. Each associated single decrement environment has its time-until-decrement random variable $T_j(x)$ and its survival function $s_{T_j}(t) = \Pr\{T_j(x) > t\}$, $j = 1, 2$. The joint survival function of $T_1(x)$ and $T_2(x)$ is given by

$$s_{T_1,T_2}(t_1, t_2) = \Pr\{(T_1(x) > t_1) \cap (T_2(x) > t_2)\}.$$

In this context, the time-until-decrement random variable T equals the minimum

of $T_1(x)$ and $T_2(x)$ and, in accordance with Section 9.3, (9.3.1), its survival function is

$$s_T(t) = s_{T_1,T_2}(t, t).$$

If $T_1(x)$ and $T_2(x)$ are independent,

$$s_T(t) = s_{T_1}(t)s_{T_2}(t) = s_{T_1,T_2}(t, 0)s_{T_1,T_2}(0, t),$$

and

$$\begin{aligned}\mu_x^{(\tau)}(t) &= -\frac{d}{dt}\log s_{T_1}(t)s_{T_2}(t) \\ &= \mu_x^{(1)}(t) + \mu_x^{(2)}(t). \end{aligned} \tag{10.5.13}$$

On the other hand, if $T_1(x)$ and $T_2(x)$ are dependent,

$$\begin{aligned}\mu_x^{(\tau)}(t) &= -\frac{d}{dt}\log s_{T_1,T_2}(t, t) \\ &\neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t). \end{aligned} \tag{10.5.14}$$

The two terms on the right-hand side of (10.5.13) are called ***marginal forces of decrement*** associated, in order, with $T_1(x)$ and $T_2(x)$.

If $T_1(x)$ and $T_2(x)$ are independent, then the marginal forces of decrement from a single decrement environment can be used with (10.5.2) to determine ${}_tp_x^{(\tau)}$. If $T_1(x)$ and $T_2(x)$ are dependent, we have no assurance that assuming (10.5.2) yields the survival function of time-until-decrement in a multiple decrement environment.

Example 10.5.2

This example builds on Examples 9.2.1, 9.2.2, and 9.3.1. The dependent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$\begin{aligned} f_{T_1,T_2}(s, t) &= 0.0006(s - t)^2 \qquad 0 < s < 10,\ 0 < t < 10 \\ &= 0 \qquad\qquad\qquad\quad \text{elsewhere.} \end{aligned}$$

The joint survival function is exhibited in Example 9.2.2, and the survival function of $T = \min[T_1, T_2]$ is exhibited in Example 9.3.1.

Show that

$$-\frac{d}{dt}\log s_T(t) \neq -\frac{d}{dt}\log s_{T_1,T_2}(t, 0) - \frac{d}{dt}\log s_{T_1,T_2}(0, t).$$

Solution:

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{(10 - t)} \qquad 0 < t < 10,$$

and

$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{4{,}000 - 1{,}200t + 120t^2}{20{,}000 - 4{,}000t + 600t^2 - 40t^3}$$
$$= \frac{100 - 30t + 3t^2}{500 - 100t + 15t^2 - t^3}$$
$$= \frac{100 - 30t + 3t^2}{(10 - t)(50 - 5t + t^2)},$$

which by symmetry is also equal to $-\frac{d}{dt}\log s_{T_1,T_2}(0,t)$. Therefore

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10 - t} \neq \frac{1}{10 - t}\left(\frac{200 - 60t + 6t^2}{50 - 5t + t^2}\right).$$

▼

Example 10.5.3

This example builds on Examples 9.2.3 and 9.3.2. The independent random variables $T_1(x)$ and $T_2(x)$ have a joint p.d.f. given by

$$f_{T_1,T_2}(s,t) = [0.02(10 - s)][0.02(10 - t)] \qquad \begin{matrix} 0 < s < 10 \\ 0 < t < 10. \end{matrix}$$

Show that

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1,T_2}(t,0) - \frac{d}{dt}\log s_{T_1,T_2}(0,t).$$

Solution:

The survival function of $T = \min[T_1(x), T_2(x)]$ is displayed in Example 9.3.2. Therefore,

$$-\frac{d}{dt}\log s_T(t) = \frac{4}{10 - t} \qquad 0 < t < 10,$$

$$-\frac{d}{dt}\log s_{T_1,T_2}(t,0) = \frac{2}{10 - t} \qquad 0 < t < 10,$$

and by symmetry

$$-\frac{d}{dt}\log s_{T_1,T_2}(0,t) = \frac{2}{10 - t} \qquad 0 < t < 10.$$

As a result,

$$-\frac{d}{dt}\log s_T(t) = -\frac{d}{dt}\log s_{T_1T_2}(t,0) - \frac{d}{dt}\log s_{T_1T_2}(0,t).$$

▼

An interesting but distressing aspect of Examples 10.5.2 and 10.5.3 is that two dependent time-until-decrement random variables and two independent time-until-decrement random variables yield the same distribution of $T = \min[T_1(x), T_2(x)]$. Values of T can be observed, but without additional information it is impossible

to select between the two models that may be generating the data. This, as in Section 9.3, is an example of nonidentifiability. Henceforth, in this chapter when constructing multiple decrement distributions from associated single decrement distributions, we assume that the component random variables are independent.

Remark:

The correspondence between the theory for the joint life model and the theory for the multiple decrement model can provide insights, but it is not complete. The difference between the two models centers on two facts that were identified in the discussion of (10.5.2) and Example 10.5.2. Realizations of both $T(x)$ and $T(y)$ can, at least in theory, be observed, while only the minimum of $T_1(x)$ and $T_2(x)$ and which one is the minimum can be observed. The corresponding problem in estimating joint life models was mentioned in Section 9.3. In addition, $\lim_{t\to\infty} {}_tp_x = \lim_{t\to\infty} {}_tp_y = 0$, whereas there is no assurance that

$$\lim_{t\to\infty} {}_tp'^{(j)}_x = 0 \qquad j = 1, 2.$$

10.6 Construction of a Multiple Decrement Table

In building a multiple decrement model it is best if data, including that on age and cause of decrement for the population under study, can be used to estimate directly the probabilities $q_x^{(j)}$. Large, well-established employee benefit plans may have such data. For other plans, such data are frequently not available. An alternative is to construct the model from associated single decrement rates assumed appropriate for the population under study. The adequacy of the model should then be tested by reviewing data as they become available.

Once satisfactory associated single decrement tables are selected, the results of Section 10.5 can be used to complete the construction of the multiple decrement table. The availability of a set of $p'^{(j)}_x$, for $j = 1, 2, \ldots, m$ and all values of x, will permit the computation of $p_x^{(\tau)}$ by (10.5.2) and of $q_x^{(\tau)}$ by $q_x^{(\tau)} = 1 - p_x^{(\tau)}$. The remaining step is to break $q_x^{(\tau)}$ into its components $q_x^{(j)}$ for $j = 1, 2, \ldots, m$. If either the constant force or the uniform distribution of decrement assumption is adopted in the model, (10.5.9) can be used for the calculation of the $q_x^{(j)}$.

Example 10.6.1

Use (10.5.2) and (10.5.9) to obtain the multiple decrement table corresponding to absolute rates of decrement given below. Presumably the actuary has examined the characteristics of the participant group and has decided that associated single decrement tables yielding these rates are appropriate for the group under study. It is also assumed that cause 3 is retirement that can occur between ages 65 and 70 and is mandatory at 70.

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
65	0.020	0.02	0.04
66	0.025	0.02	0.06
67	0.030	0.02	0.08
68	0.035	0.02	0.10
69	0.040	0.02	0.12

Solution:

The table below contains the results of the calculation of the probabilities of decrement. Formula (10.5.2) can be rewritten as

$$q_x^{(\tau)} = 1 - \prod_{j=1}^{3} (1 - q_x'^{(j)}).$$

In this equation the assumed independence among the three causes of decrement is apparent. Formula (10.5.9), and the mandatory retirement condition, yield the multiple decrement probabilities. The multiple decrement table is constructed as in Example 10.3.1.

x	$q_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
65	0.07802	0.01940	0.01940	0.03921	1 000.00	19.40	19.40	39.21
66	0.10183	0.02401	0.01916	0.05867	921.99	22.14	17.67	54.09
67	0.12545	0.02851	0.01891	0.07803	828.09	23.61	15.66	64.62
68	0.14887	0.03290	0.01866	0.09731	724.20	23.83	13.51	70.47
69	0.17210	0.03720	0.01841	0.11649	616.39	22.93	11.35	71.80
70	1.00000	0.00000	0.00000	1.00000	510.31	0.00	0.00	510.31

▼

It has been noted that (10.5.9) and (10.5.10) will not be used if $p_x'^{(j)}$ or $p_x^{(\tau)} = 0$; some alternative device will be necessary. One such method, which handles this indeterminacy and lends itself to special adjustments, is based on assumed distributions of decrement in the associated single decrement tables rather than on assumptions about multiple decrement probabilities as in Section 10.5. We first examine an assumption of uniform distribution of decrement (in each year of age) in the associated single decrement tables. We restrict our attention to situations with three decrements, but the method and formulas easily extend for $m > 3$. Under the stated assumption,

$$ {}_tp_x'^{(j)} = 1 - t\, q_x'^{(j)} \qquad j = 1, 2, 3;\ 0 \le t \le 1 \tag{10.6.1}$$

and

$$ {}_tp_x'^{(j)}\, \mu_x^{(j)}(t) = \frac{d}{dt}\left(-{}_tp_x'^{(j)}\right) = q_x'^{(j)}. \tag{10.6.2}$$

It follows that

$$q_x^{(1)} = \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt$$

$$= \int_0^1 {}_tp_x'^{(1)}\, \mu_x^{(1)}(t)\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)}\, dt$$

$$= q_x'^{(1)} \int_0^1 (1 - t\, q_x'^{(2)})(1 - t\, q_x'^{(3)})\, dt$$

$$= q_x'^{(1)} \left[1 - \frac{1}{2}(q_x'^{(2)} + q_x'^{(3)}) + \frac{1}{3} q_x'^{(2)} q_x'^{(3)} \right]. \qquad (10.6.3)$$

Similar formulas hold for $q_x^{(2)}$, $q_x^{(3)}$, and it can be verified that

$$\begin{aligned} q_x^{(1)} + q_x^{(2)} + q_x^{(3)} &= q_x'^{(1)} + q_x'^{(2)} + q_x'^{(3)} \\ &\quad - (q_x'^{(1)} q_x'^{(2)} + q_x'^{(1)} q_x'^{(3)} + q_x'^{(2)} q_x'^{(3)}) \\ &\quad + q_x'^{(1)} q_x'^{(2)} q_x'^{(3)} \\ &= 1 - (1 - q_x'^{(1)})(1 - q_x'^{(2)})(1 - q_x'^{(3)}) = q_x^{(\tau)}. \qquad (10.6.4) \end{aligned}$$

Example 10.6.2

Obtain the probabilities of decrement for ages 65–69 from the data in Example 10.6.1, under the assumption of a uniform distribution of decrement in each year of age in each of the associated single decrement tables.

Solution:

This is an application of (10.6.3).

x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
65	0.020	0.02	0.04	0.01941	0.01941	0.03921
66	0.025	0.02	0.06	0.02401	0.01916	0.05866
67	0.030	0.02	0.08	0.02852	0.01892	0.07802
68	0.035	0.02	0.10	0.03292	0.01867	0.09727
69	0.040	0.02	0.12	0.03723	0.01843	0.11643

These probabilities are close to those obtained by (10.5.9), displayed in Example 10.6.1. ▼

We conclude this section with another example illustrating the use of a special distribution for one of the decrements. Special distributions are sometimes required by the facts of the situation being modeled.

Example 10.6.3

Consider a situation with three causes of decrement: mortality, disability, and withdrawal. Assume mortality and disability are uniformly distributed in each year of age in the associated single decrement tables with absolute rates of $q_x'^{(1)}$ and

$q_x'^{(2)}$, respectively. Also assume that withdrawals occur only at the end of the year with an absolute rate of $q_x'^{(3)}$.

a. Give formulas for the probabilities of decrement in the year of age x to $x + 1$ for the three causes.

b. Reformulate the probabilities under the assumptions that
   (i) In the associated single decrement model, withdrawals occur only at the age's midyear or year end, and
   (ii) Equal proportions, namely (1 / 2), $q_x'^{(3)}$, of those beginning the year withdraw at the midyear and at the year end.

Remark:

Until now our multiple decrement models have been fully continuous, except possibly to recognize a mandatory retirement age. Moreover, our theory began with a multiple decrement model and after defining the forces $\mu_x^{(j)}(t)$, $j = 1, 2, \ldots, m$ proceeded to the associated single decrement tables. In this example we start with the single decrement tables, and in one of these tables the decrement takes place discretely at the ends of stated intervals. We do not attempt to define a force of decrement for this discrete case but proceed by direct methods to build, from the single decrement tables, a multiple decrement model possessing the relationships (10.2.19) and (10.5.2) established in our prior theory.

Solution:

a. Figure 10.6.1 displays survival factors for the given single decrement tables and for a multiple decrement table where

$$ {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)} $$

for nonintegral $t \geq 0$. At $t = 1$, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ are discontinuous, so we consider

$$ \lim_{t \to 1-} {}_tp_x^{(\tau)} = {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, 1 $$

and

$$ p_x^{(\tau)} = p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

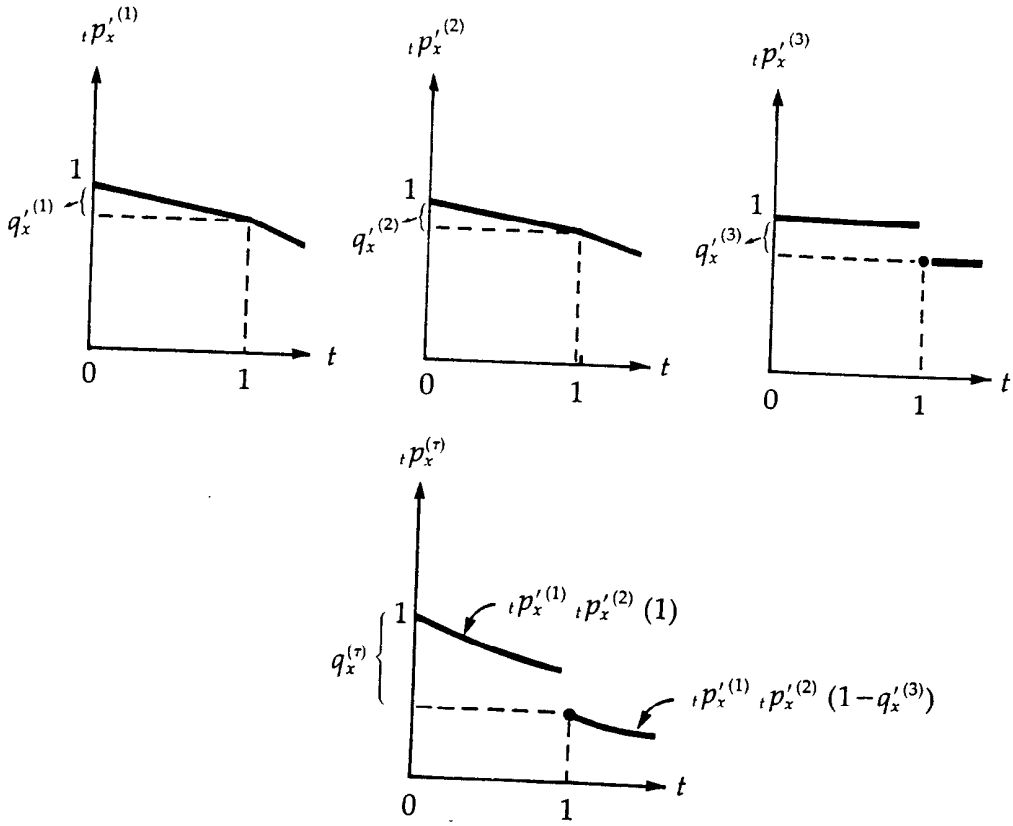
We also require that, for our multiple decrement table,

$$ q_x^{(\tau)} = q_x^{(1)} + q_x^{(2)} + q_x^{(3)} = 1 - p_x^{(\tau)} = 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}). $$

We set

$$ \begin{aligned} q_x^{(1)} &= \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt \\ &= \int_0^1 {}_tp_x'^{(1)}\, {}_tp_x'^{(2)} (1) \mu_x^{(1)}(t)\, dt \\ &= q_x'^{(1)} \int_0^1 (1 - t q_x'^{(2)})\, dt \\ &= q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(2)}\right). \end{aligned} $$

Survival Factors, ${}_tp_x'^{(j)}$, $j = 1, 2, 3$, and ${}_tp_x^{(\tau)}$



Similarly, we set

$$q_x^{(2)} = q_x'^{(2)}\left(1 - \frac{1}{2}\,q_x'^{(1)}\right).$$

Then

$$\begin{aligned} q_x^{(3)} &= q_x^{(\tau)} - (q_x^{(1)} + q_x^{(2)}) \\ &= 1 - p_x'^{(1)}p_x'^{(2)}(1 - q_x'^{(3)}) - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)}, \end{aligned}$$

and, since

$$1 - q_x'^{(1)} - q_x'^{(2)} + q_x'^{(1)}q_x'^{(2)} = p_x'^{(1)}p_x'^{(2)},$$

$$q_x^{(3)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)}.$$

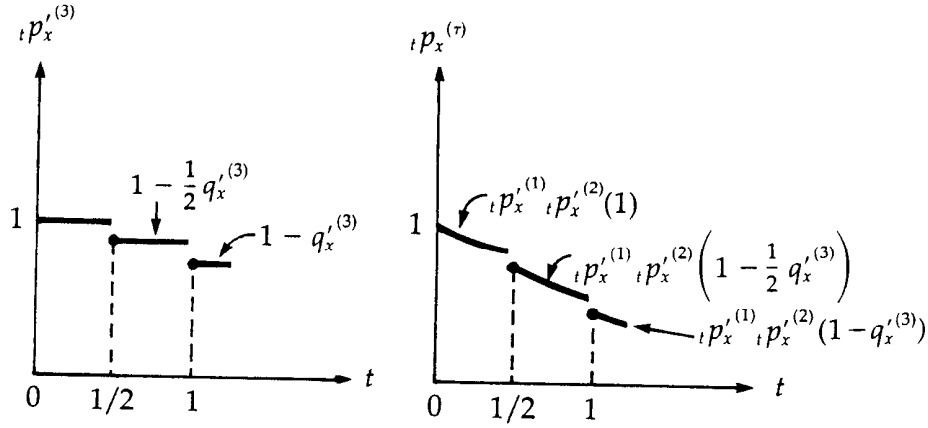
Note that

$$\lim_{t\to 1-} {}_tp_x^{(\tau)} - \lim_{t\to 1+} {}_tp_x^{(\tau)} = p_x'^{(1)}p_x'^{(2)}q_x'^{(3)} = q_x^{(3)};$$

that is, the discontinuity at $t = 1$ equals $q_x^{(3)}$.

b. Here ${}_tp_x'^{(1)}$ and ${}_tp_x'^{(2)}$ are as in Figure 10.6.1, but ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$ now have discontinuities at $t = 1/2$ and $t = 1$, as shown in Figure 10.6.2.

Survival Factors, ${}_tp_x'^{(3)}$ and ${}_tp_x^{(\tau)}$



Proceeding as in (a), but taking account of the intervals $[0, 1/2)$ and $[1/2, 1)$, we set

$$\begin{aligned} q_x^{(1)} &= q_x'^{(1)} \int_0^{1/2} (1 - t\, q_x'^{(2)})\, dt \\ &\quad + q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(3)}\right) \int_{1/2}^{1} (1 - t\, q_x'^{(2)})\, dt \\ &= q_x'^{(1)} \left(1 - \frac{1}{2} q_x'^{(2)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(2)} q_x'^{(3)}\right). \end{aligned}$$

Similarly, we set

$$q_x^{(2)} = q_x'^{(2)} \left(1 - \frac{1}{2} q_x'^{(1)} - \frac{1}{4} q_x'^{(3)} + \frac{3}{16} q_x'^{(1)} q_x'^{(3)}\right).$$

Then,

$$\begin{aligned} q_x^{(3)} &= 1 - p_x^{(\tau)} - q_x^{(1)} - q_x^{(2)} \\ &= 1 - p_x'^{(1)} p_x'^{(2)} (1 - q_x'^{(3)}) - q_x^{(1)} - q_x^{(2)}, \end{aligned}$$

which reduces to

$$q_x^{(3)} = q_x'^{(3)} \left(1 - \frac{3}{4} q_x'^{(1)} - \frac{3}{4} q_x'^{(2)} + \frac{5}{8} q_x'^{(1)} q_x'^{(2)}\right). \quad \blacktriangledown$$

10.7 Notes and References

The history of multiple decrement theory was reviewed by Seal (1977). Chiang (1968) developed the theory using the language of competing risks. The foundation for the actuarial theory of multiple decrement models was built by Makeham (1874). Menge (1932) and Nesbitt and Van Eenam (1948) provided insight into the deterministic interpretation of forces of decrement and of increment. Bicknell and

Nesbitt (1956) developed a very general theory for individual insurances using a deterministic multiple decrement model. Hickman (1964) redeveloped this theory using the language of the stochastic model, and this redevelopment is the basis for much of this chapter. The analysis of life tables by cause of death is the subject of papers by Greville (1948) and Preston, Keyfitz, and Schoen (1973).

The perplexing estimation issues that arise when the times-until-decrement are not independent are discussed by Elandt-Johnson and Johnson (1980). Promislow (1991b) makes the excellent point that in practice multiple decrement models should be select in the sense of Chapter 3. He developed a theory and associated notation for select multiple decrement models. Exercises 10.3 and 10.24 are built on a discussion by Robinson (1984).

Carriere (1994) applied copulas, discussed in Section 9.6.2, to create multiple decrement distributions that incorporate dependent component random variables. Carriere also discusses the problem of identifiability and reviews the conditions under which it is possible to identify a unique joint survival function $s_{T_1(x)\ T_2(x)}(t, t)$.

Exercises

Section 10.2

10.1. Let $\mu_x^{(j)}(t) = \mu_x^{(j)}(0)$, $j = 1, 2, \ldots, m$, $t \geq 0$. Obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_J(j)$ c. $f_T(t)$.
The functions called for in (a) and (c) are p.d.f.'s, and the function in (b) is a p.f. Show that T and J are independent random variables.

10.2. A multiple decrement model with two causes of decrement has forces of decrement given by

$$\mu_x^{(1)}(t) = \frac{1}{100 - (x + t)}$$

and

$$\mu_x^{(2)}(t) = \frac{2}{100 - (x + t)} \qquad t < 100 - x.$$

If $x = 50$, obtain expressions for
a. $f_{T,J}(t, j)$ b. $f_T(t)$ c. $f_J(j)$ d. $f_{J|T}(j|t)$.

10.3. Given the joint p.d.f.

$$f_{T,J}(t, j) = pu_1\, e^{-(u_1+v_1)t} + (1 - p)u_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t,\ j = 1$$

$$= pv_1\, e^{-(u_1+v_1)t} + (1 - p)v_2\, e^{-(u_2+v_2)t} \qquad 0 \leq t,\ j = 2$$

where $0 < p < 1$ and $0 < u_1, u_2, v_1, v_2,$

find

a. The marginal p.d.f.s $f_T(t)$ and $f_J(j)$
b. The survival function $s_T(t)$.

Section 10.3

10.4. Using the multiple decrement probabilities given in Example 10.3.1, evaluate the following:

a. ${}_3p_{65}^{(\tau)}$ b. ${}_{3|}q_{65}^{(1)}$ c. ${}_3q_{65}^{(2)}$.

10.5. The following multiple decrement probabilities apply to students entering a 4-year college.

Curtate Duration, at Beginning of Academic Year	Probability of		
	Academic Failure, $j = 1$	Withdrawal for All Other Reasons, $j = 2$	Survival through the Academic Year
0	0.15	0.25	0.60
1	0.10	0.20	0.70
2	0.05	0.15	0.80
3	0.00	0.10	0.90

An entering class has 1,000 members.

a. What is the expectation of the number of graduates? What is the variance?
b. What is the expected number of those who will fail sometime during the 4-year program? What is the variance of the number of students who will fail?

10.6. Construct a multiple decrement table on the basis of the data in Exercise 10.5 and use it to exhibit

a. The marginal distribution of the random variable J (mode of exit), which takes on values for academic failure, withdrawal, and graduation
b. The conditional distribution of the mode of termination, given that a student has terminated in the third year.

Section 10.4

10.7. Given that $\mu^{(1)}(x) = 1/(a - x)$, $0 \le x < a$, and $\mu^{(2)}(x) = 1$, derive expressions for

a. $l_x^{(\tau)}$ b. $d_x^{(1)}$ c. $d_x^{(2)}$.

Assume $l_0^{(\tau)} = a$.

10.8. Given $\mu^{(1)}(x) = 2x/(a - x^2)$, $0 \le x \le \sqrt{a}$, and $\mu^{(2)}(x) = c$, $c > 0$, and $l_0^{(\tau)} = 1{,}000$, derive an expression for $l_x^{(\tau)}$.

10.9. Derive expressions for the following derivatives:

a. $\frac{d}{dx}\,{}_tq_x^{(\tau)}$ b. $\frac{d}{dx}\,{}_tq_x^{(j)}$ c. $\frac{d}{dt}\,{}_tq_x^{(j)}$.

Section 10.5

10.10. Using the data in Exercise 10.5, and assuming a uniform distribution of all decrements in the multiple decrement model, calculate a table of $q_k'^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration).

10.11. If $\mu_x^{(1)}(t)$ is a constant c for $0 \leq t \leq 1$, derive expressions in terms of c and ${}_tp_x^{(\tau)}$ for

a. $q_x'^{(1)}$ b. $m_x^{(1)}$ c. $q_x^{(1)}$.

10.12. Show that under appropriate assumptions of a uniform distribution of decrements

a. $m_x^{(\tau)} = \dfrac{q_x^{(\tau)}}{1 - (1/2)\, q_x^{(\tau)}}$ b. $m_x^{(j)} = \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}$ c. $m_x'^{(j)} = \dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}}$

and, conversely,

d. $q_x^{(\tau)} = \dfrac{m_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$ e. $q_x^{(j)} = \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}$ f. $q_x'^{(j)} = \dfrac{m_x'^{(j)}}{1 + (1/2)\, m_x'^{(j)}}$.

10.13. Order the following in terms of magnitude and state your reasons:

$$q_x'^{(j)}, \quad q_x^{(j)}, \quad m_x'^{(j)}.$$

10.14. Given, for a double decrement table, that $q_{40}'^{(1)} = 0.02$ and $q_{40}'^{(2)} = 0.04$, calculate $q_{40}^{(\tau)}$ to four decimal places.

10.15. For a double decrement table you are given that $m_{40}^{(\tau)} = 0.2$ and $q_{40}'^{(1)} = 0.1$. Calculate $q_{40}'^{(2)}$ to four decimal places assuming

a. Uniform distribution of decrements in the multiple decrement model
b. Uniform distribution of decrements in the associated single decrement tables.

10.16. Using the data in Exercise 10.5 and assuming a uniform distribution of decrements in the multiple decrement model, construct a table of $m_k^{(j)}$, $j = 1, 2$, $k = 0, 1, 2, 3$ (where k is the curtate duration). Calculate each result to five decimal places.

10.17. Given that decrement may be due to death, 1, disability, 2, or retirement, 3, use (10.5.9) to construct a multiple decrement table based on the following absolute rates.

Age x	$q_x'^{(1)}$	$q_x'^{(2)}$	$q_x'^{(3)}$
62	0.020	0.030	0.200
63	0.022	0.034	0.100
64	0.028	0.040	0.120

10.18. Recalculate the multiple decrement table from the absolute rates of decrement in Exercise 10.17 by means of the ***central rate bridge.*** [Hint: To use the

central rate bridge, first calculate $m_x'^{(j)}$ by the formula

$$m_x'^{(j)} \cong \frac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \qquad j = 1, 2, 3,$$

which holds if there is a uniform distribution of decrement in the associated single decrement tables. Next, assume $m_x^{(j)} \cong m_x'^{(j)}$, $j = 1, 2, 3$, and proceed to $q_x^{(j)}$ by

$$q_x^{(j)} = \frac{d_x^{(j)}}{l_x^{(\tau)}} = \frac{d_x^{(j)}}{l_x^{(\tau)} - (1/2)\, d_x^{(\tau)} + (1/2)\, d_x^{(\tau)}} = \frac{m_x^{(j)}}{1 + (1/2)\, m_x^{(\tau)}}.$$

This second relation holds if there is a uniform distribution of total decrement in the multiple decrement table. But then

$$\begin{aligned} {}_tp_x^{(\tau)} = 1 - {}_tq_x^{(\tau)} &\neq {}_tp_x'^{(1)}\, {}_tp_x'^{(2)}\, {}_tp_x'^{(3)} \\ &= (1 - {}_tq_x'^{(1)})(1 - {}_tq_x'^{(2)})(1 - {}_tq_x'^{(3)}) \end{aligned}$$

under the condition of a uniform distribution in the associated single decrement tables. Thus there is an inconsistency in the stated conditions, but the calculations may be accurate enough for this purpose.]

10.19. Indicate arguments for the following relations:

a. $m_x'^{(j)} \cong m_x^{(j)}$

b. $\dfrac{q_x'^{(j)}}{1 - (1/2)\, q_x'^{(j)}} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\, q_x^{(\tau)}}$.

Show that these lead to

c. $q_x^{(j)} \cong \dfrac{q_x'^{(j)}\,[1 - (1/2)\, q_x^{(\tau)}]}{1 - (1/2)\, q_x'^{(j)}}$

d. $q_x'^{(j)} \cong \dfrac{q_x^{(j)}}{1 - (1/2)\,(q_x^{(\tau)} - q_x^{(j)})}$.

Compare (c) and (d) to (10.5.9) and (10.5.10).

10.20. Use the values of $q_x^{(j)}$, $q_x'^{(j)}$ from Example 10.5.1 to calculate values of $m_x^{(j)}$, $m_x'^{(j)}$, $j = 1, 2$, $x = 65, \ldots, 69$, under appropriate assumptions of uniform distribution of decrements (see Exercise 10.12).

10.21. Which of the following statements would you accept? Revise where necessary.

a. $q_x^{(j)} \cong \dfrac{m_x^{(j)}}{1 + (1/2)\, m_x^{(j)}}$

b. $\displaystyle\int_0^1 l_{x+t}^{(\tau)}\, dt \cong \frac{l_x^{(\tau)}}{1 + (1/2)\, m_x^{(\tau)}}$

c. $q_x^{(1)} = q_x'^{(1)}[1 - (1/2) q_x'^{(2)}]$ in a double decrement table where there is a

uniform distribution of decrement for the year of age x to $x + 1$ in each of the associated single decrement tables.

10.22. a. For a certain age x, particular cause of decrement j, and constant K_j, show that the following conditions are equivalent:

(i) ${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1$

(ii) $\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t \le 1$

(iii) $1 - {}_tq_x'^{(j)} = (1 - {}_tq_x^{(\tau)})^{K_j} \qquad 0 \le t \le 1.$

[Hint: Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).]

b. Verify that, in a multiple decrement table, where either

$$\mu_x^{(j)}(t) = \mu_x^{(j)}(0) \qquad 0 \le t \le 1, \, j = 1, 2, \ldots, m$$

(the constant force assumption for each cause of decrement) or

$${}_tq_x^{(j)} = t \, q_x^{(j)} \qquad 0 \le t \le 1, \, j = 1, 2, \ldots, m$$

(the uniform distribution for each cause of decrement), then

$${}_tq_x^{(j)} = K_j \, {}_tq_x^{(\tau)} \qquad 0 \le t \le 1, \, j = 1, 2, \ldots, m.$$

c. Assume that in part (a), condition (ii), $\mu_x^{(\tau)}(t)$, $0 \le t \le 1$, is given by

(i) $kt^n \qquad k > 0, \, n > 0 \qquad$ (Weibull)

(ii) $Bc^t \qquad B > 0, \, c > 1 \qquad$ (Gompertz)

and for each example find the corresponding expressions for ${}_tq_x^{(j)}$ and $1 - {}_tq_x'^{(j)}$.

10.23. a. Prove that

$$\mu_x^{(j)}(t) = K_j \, \mu_x^{(\tau)}(t) \qquad 0 \le t, \, j = 1, 2$$

where

$$K_j = \int_0^\infty {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt \qquad j = 1, 2,$$

if and only if the random variables T and J are independent.

b. If $T_1(x)$ and $T_2(x)$ are independent and J and T are independent, show that

$${}_tp_x'^{(j)} = ({}_tp_x^{(\tau)})^{K_j} \qquad j = 1, 2.$$

[Remark: Note that $K_j = f_J(j)$.]

10.24. This exercise is a continuation of Exercise 10.3 and uses the notation of Section 10.5.5. The joint survival function is given by

$$s_{T_1, T_2}(t_1, t_2) = pe^{-u_1 t_1 - v_1 t_2} + (1 - p)e^{-u_2 t_1 - v_2 t_2}$$

$$0 \le t_1, \, t_2, \, u_1, \, u_2, \, v_1, \, v_2$$

$$0 < p < 1.$$

Confirm that

$$s_{T_1, T_2}(t, t) \ne S_{T_1, T_2}(t, 0) \, S_{T_1, T_2}(0, t)$$

and

$$-\left.\frac{\partial \log s_{T_1,T_2}(t_1, t_2)}{\partial t_1}\right|_{t_1=t_2=t} \neq -\frac{d \log s_{T_1,T_2}(t, 0)}{dt}.$$

Section 10.6

10.25. Redo Exercise 10.10 by use of the formula for $q_x'^{(j)}$ in Exercise 10.19.

10.26. Show that $\mu_x^{(j)}(1/2) = m_x^{(j)}$, under the assumption of a uniform distribution of each decrement in each year of age in a multiple decrement context.

10.27. How would you proceed to construct the multiple decrement table if the given rates were those given below?
a. $q_x'^{(1)}$, $q_x'^{(2)}$, $q_x^{(3)}$
b. $q_x'^{(1)}$, $q_x^{(2)}$, $q_x^{(3)}$

10.28. In Example 10.6.2 suppose that decrement 3 at age 69 is not uniformly distributed but follows the pattern

$${}_tp_{69}'^{(3)} = \begin{cases} 1 - 0.12t & 0 < t < 1 \\ 0 & t = 1. \end{cases}$$

In words, the cause 3 absolute rate is 0.12 during the year. Then, just before age 70, all remaining survivors terminate due to cause 3. This is consistent with an assumption that $q_{69}'^{(3)} = 1$. What then is the value of $q_{69}^{(3)}$?

10.29. In a double decrement table where cause 1 is death and cause 2 is withdrawal, it is assumed that
- Deaths in the year from age h to $h + 1$ are uniformly distributed,
- Withdrawals in the year from age h to age $h + 1$ occur immediately after the attainment of age h.

From this table it is noted that, at age 50, $l_{50}^{(\tau)} = 1{,}000$, $q_{50}^{(2)} = 0.2$, and $d_{50}^{(1)} = 0.06\, d_{50}^{(2)}$. Determine $q_{50}'^{(1)}$.

Miscellaneous

10.30. On the basis of a triple decrement table, display an expression for the probability that (20) will not terminate before age 65 for cause 2.

10.31. a. You are given $q_x'^{(1)}$, $q_x'^{(2)}$, $m_x^{(3)}$, $m_x^{(4)}$. How would you proceed to construct a multiple decrement table where active service of an employee group is subject to decrement from death, 1, withdrawal, 2, disability, 3, and retirement, 4?
b. On the basis of the table in (a), give an expression for the probability that, in the future, an active member age y will not retire but will terminate from service for some other cause.

10.32. Prove and interpret the relation

$$q_x^{(j)} = q_x'^{(j)} - \sum_{k \neq j} \int_0^1 {}_tp_x^{(\tau)}\, \mu_x^{(k)}(t)\, {}_{1-t}q_{x+t}'^{(j)}\, dt.$$

10.33. Let

$$w^{(\tau)}(t) = \frac{{}_tp_x^{(\tau)}}{\int_0^1 {}_tp_x^{(\tau)}\,dt}$$

and

$$w^{(j)}(t) = \frac{{}_tp_x'^{(j)}}{\int_0^1 {}_tp_x'^{(j)}\,dt} \quad 0 \le t \le 1.$$

Assume that j and at least one other cause have positive forces of decrement on the interval $0 \le t \le 1$.

a. Show that
   (i) $w^{(\tau)}(0) > w^{(j)}(0)$
   (ii) $w^{(\tau)}(1) < w^{(j)}(1)$
   (iii) There exists a unique number r, $0 < r < 1$, such that $w^{(\tau)}(r) = w^{(j)}(r)$.

b. Let

$$-I = \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

Show that

$$I = \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]dt.$$

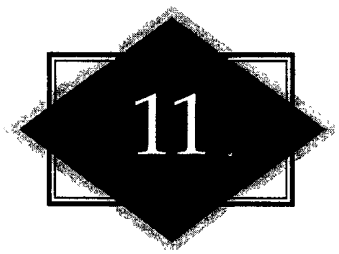
c. Assume that $\mu_x^{(j)}(t)$ is an increasing function on the interval $0 \le t \le 1$. Use the mean value theorem for integrals to establish the following inequalities:

$$\begin{aligned} m_x'^{(j)} - m_x^{(j)} &= \int_0^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &= \int_0^r [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &\quad + \int_r^1 [w^{(j)}(t) - w^{(\tau)}(t)]\mu_x^{(j)}(t)\,dt \\ &= -\mu_x^{(j)}(t_0)\,I + \mu_x^{(j)}(t_1)\,I \qquad 0 < t_0 < r < t_1 < 1 \\ &= I\,[\mu_x^{(j)}(t_1) - \mu_x^{(j)}(t_0)] > 0. \end{aligned}$$

10.34. The joint distribution of T and J is specified by

$$\left.\begin{aligned}\mu_x^{(1)}(t) &= \frac{\theta t^{\alpha-1} e^{-\beta t}}{\int_t^\infty s^{\alpha-1} e^{-\beta s} ds} \\ \mu_x^{(2)}(t) &= \frac{(1-\theta) t^{\alpha-1} e^{-\beta t}}{\int_t^\infty s^{\alpha-1} e^{-\beta s} ds}\end{aligned}\right\} \begin{array}{l} 0 < \theta < 1 \\ \alpha > 0 \\ \beta > 0 \\ t \geq 0. \end{array}$$

a. Obtain expressions for $f_{T,J}(t, j)$, $f_J(j)$, and $f_T(t)$.
b. Express E[T] and Var(T) in terms of α and β.
c. Confirm that J and T are independent.



APPLICATIONS OF MULTIPLE DECREMENT THEORY

11.1 Introduction

The multiple decrement model developed in Chapter 10 provides a framework for studying many financial security systems. For example, life insurance policies frequently provide for special benefits if death occurs by accidental means or if the insured becomes disabled. The single decrement model, the subject of Chapters 3 through 9, does not provide a mathematical model for policies with such multiple benefits. In addition, there may be nonforfeiture benefits that are paid when the insured withdraws from the set of premium-paying policyholders. The determination of the amount of these nonforfeiture benefits, and related public policy issues, are discussed in Chapter 16. The basic models associated with these multiple benefits are developed in this chapter.

Another major application of multiple decrement models is in pension plans. In this chapter we consider basic methods used in calculating the actuarial present values of benefits and contributions for a participant in a pension plan. The participants of a plan may be a group of employees of a single employer, or they may be the employees of a group of employers engaged in similar activities. A plan, upon a participant's retirement, typically provides pensions for age and service or for disability. In case of withdrawal from employment, there can be a return of accumulated participant contributions or a deferred pension. For death occurring before the other contingencies, there can be a lump sum or income payable to a beneficiary. Payments to meet the costs of the benefits are referred to as contributions, not premiums as for insurance, and are payable in various proportions by the participants and the plan sponsor.

A pension plan can be regarded as a system for purchasing deferred life annuities (payable during retirement) and certain ancillary benefits with a temporary annuity of contributions during active service. The balancing of the actuarial present values of benefits and contributions may be on an individual basis, but frequently it is on

some aggregate basis for the whole group of participants. Methods to accomplish this balance comprise the theory of pension funding. Here we are concerned with only the separate valuation of the pension plan's actuarial present value of benefits and contributions with respect to a typical participant. Aggregate values can then be obtained by summation over all the participants. The basic tools for valuing the benefits of, and the contributions to, a pension plan are presented here, but their application to the possible funding methods for a plan is deferred to Chapter 20.

In Section 11.6 we study disability benefits commonly found in conjunction with individual life insurance. The benefits include those for waiver of premium and for disability income. There is a discussion of a widely used single decrement approximation for calculating benefit premiums and benefit reserves for these disability coverages.

11.2 Actuarial Present Values and Their Numerical Evaluation

Actuarial applications of multiple decrement models arise when the amount of benefit payment depends on the mode of exit from the group of active lives. We let $B_{x+t}^{(j)}$ denote the value of a benefit at age $x + t$ incurred by a decrement at that age by cause j. Then the actuarial present value of the benefits, denoted in general by $\bar{A}$, will be given by

$$\bar{A} = \sum_{j=1}^{m} \int_0^{\infty} B_{x+t}^{(j)} \, v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t) \, dt. \tag{11.2.1}$$

If $m = 1$ and $B_{x+t}^{(1)} = 1$, $\bar{A}$ reduces to $\bar{A}_x$, the actuarial present value for a unit of whole life insurance with immediate payment of claims.

More appropriate for this chapter is the example of a ***double indemnity provision,*** which provides for the death benefit to be doubled when death is caused by accidental means. Let $J = 1$ for death by accidental means and $J = 2$ for death by other means, and take $B_{x+t}^{(1)} = 2$ and $B_{x+t}^{(2)} = 1$. The actuarial present value for an n-year term insurance is given by

$$\bar{A} = 2 \int_0^n v^t \, {}_tp_x^{(\tau)} \mu_x^{(1)}(t) \, dt + \int_0^n v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(2)}(t) \, dt. \tag{11.2.2}$$

For numerical evaluation, the first step is to break the expression into a set of integrals, one for each of the years involved. For the first integral,

$$2 \int_0^n v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t) \, dt = 2 \sum_{k=0}^{n-1} v^k \, {}_kp_x^{(\tau)} \int_0^1 v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(1)}(k + s) \, ds.$$

If now we assume, as for (10.5.11), that each decrement in the multiple decrement context has a uniform distribution in each year of age, we have

$$2\int_0^n v^t\ {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt = 2\sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(1)} \int_0^1 (1+i)^{1-s}\ ds$$

$$= \frac{2i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(1)}.$$

Applying a similar argument for the second integral and combining, we get

$$\bar{A} = \frac{i}{\delta}\left[\sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x^{(\tau)}(\ (2q_{x+k}^{(1)} + q_{x+k}^{(2)})\right]$$

$$= \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x^{(\tau)}\ (q_{x+k}^{(1)} + q_{x+k}^{(\tau)})$$

$$= \bar{A}^{1\,(1)}_{x:\overline{n}|} + \bar{A}^{1}_{x:\overline{n}|}, \tag{11.2.3}$$

where $\bar{A}^{1\,(1)}_{x:\overline{n}|}$ is the actuarial present value of term insurance benefits of 1 covering death from accidental means and $\bar{A}^{1}_{x:\overline{n}|}$ is the actuarial present value for term insurance benefits of 1 covering death from all causes. Here ${}_kp_x^{(\tau)}$ could be taken as the survival function from a mortality table. If values of $q_{x+k}^{(1)}$ are available, it would be unnecessary to develop the full double decrement table in order to calculate (11.2.3) under the assumption that each decrement has a uniform distribution in each year of age.

This example is simple because the benefit amount does not change as a function of age at decrement, and, in particular, it does not change within a year of age. For a contrasting example, we take $B_{x+t}^{(1)} = t$ and $B_{x+t}^{(2)} = 0$ for $t > 0$. In this case,

$$\bar{A} = \int_0^\infty t\ v^t\ {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt = \sum_{k=0}^{\infty} v^k\ {}_kp_x^{(\tau)} \int_0^1 (k+s)v^s\ {}_sp_{x+k}^{(\tau)}\ \mu_x^{(1)}(k+s)\ ds.$$

We again make the assumption that each decrement in the multiple decrement context has a uniform distribution in each year of age, and we obtain

$$\bar{A} = \sum_{k=0}^{\infty} v^{k+1}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(1)} \int_0^1 (k+s)(1+i)^{1-s}\ ds$$

$$= \sum_{k=0}^{\infty} v^{k+1}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(1)}\ \frac{i}{\delta}\left(k + \frac{1}{\delta} - \frac{1}{i}\right). \tag{11.2.4}$$

In practice, $B_{x+t}^{(j)}$ is often a complicated function, possibly requiring some degree of approximation. For such a case, if we apply the uniform distribution assumption to the j-th integral in (11.2.1), we obtain

$$\sum_{k=0}^{\infty} v^{k+1}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(j)} \int_0^1 B_{x+k+s}^{(j)}\ (1+i)^{1-s}\ ds.$$

Then, use of the midpoint integration rule yields

$$\sum_{k=0}^{\infty} v^{k+1/2}\ {}_kp_x^{(\tau)}\ q_{x+k}^{(j)}\ B_{x+k+1/2}^{(j)} \tag{11.2.5}$$

as a practical formula for the evaluation of the integral.

As an example, we return to (11.2.4) where the quantity

$$k + \frac{1}{\delta} - \frac{1}{i}$$

can be viewed as an effective mean benefit amount for the year $k + 1$, and the familiar i/δ term can be viewed as the correction needed to provide immediate payment of claims. The value given by (11.2.4) is closely approximated by

$$\sum_{k=0}^{\infty} v^{k+1/2}\, {}_kp_x^{(\tau)}\, q_{x+k}^{(j)} \left(k + \frac{1}{2}\right), \tag{11.2.6}$$

which makes use of the midpoint rule for approximate integration to evaluate

$$\int_0^1 (k + s)(1 + i)^{1-s}\, ds.$$

In Section 10.6 we discussed situations where a uniform distribution of decrement assumption was not appropriate. For such situations, special adjustments to the actuarial present value should be made. We reexamine Example 10.6.3 where, in the associated single decrement model for decrement (3), one-half the expected withdrawals occur at midyear and the other half occur at year end. The actuarial present value for withdrawal benefits is given by

$$\bar{A} = \sum_{k=0}^{\infty} v^k\, {}_kp_x^{(\tau)} \left[\frac{1}{2}\, q_{x+k}^{\prime(3)}\, v^{1/2}\, B_{x+k+1/2}^{(3)} \left(1 - \frac{1}{2}\, q_{x+k}^{\prime(1)}\right)\left(1 - \frac{1}{2}\, q_{x+k}^{\prime(2)}\right) + \frac{1}{2}\, q_{x+k}^{\prime(3)}\, v\, B_{x+k+1}^{(3)} (1 - q_{x+k}^{\prime(1)})(1 - q_{x+k}^{\prime(2)})\right].$$

Here we are dealing with the distribution of decrement in the context of the associated single decrement tables, rather than in the multiple decrement context. A possible approximation would be to use a geometric average value of the interest factor in the year of withdrawal, such as $v^{3/4}$, and the arithmetic average value of the withdrawal benefit, such as

$$\hat{B}_{x+k}^{(3)} = \frac{1}{2}\left(B_{x+k+1/2}^{(3)} + B_{x+k+1}^{(3)}\right).$$

Thus,

$$\begin{aligned}\bar{A} &\cong \sum_{k=0}^{\infty} v^{k+3/4}\, {}_kp_x^{(\tau)}\, \hat{B}_{x+k}^{(3)} \left[\frac{1}{2}\, q_{x+k}^{\prime(3)} \left(1 - \frac{1}{2}\, q_{x+k}^{\prime(1)}\right)\left(1 - \frac{1}{2}\, q_{x+k}^{\prime(2)}\right) + \frac{1}{2}\, q_{x+k}^{\prime(3)} (1 - q_{x+k}^{\prime(1)})(1 - q_{x+k}^{\prime(2)})\right] \\ &= \sum_{k=0}^{\infty} v^{k+3/4}\, {}_kp_x^{(\tau)}\, q_{x+k}^{(3)}\, \hat{B}_{x+k}^{(3)}.\end{aligned}$$

Remark:

In this section we have not used the format employed in Chapter 6 to state premium determination problems. This was done to achieve brevity. The premium problems of this section could have been approached by formulating a loss function and invoking the equivalence principle or some other premium principle.

Assume, for example, an insurance to (x) paying

a. $2B$ upon death due to an accident before age r
b. B upon death due to all other causes before age r, and
c. B upon death after age r.

Two causes of decrement are recognized, $J = 1$, the accidental cause, and $J = 2$, the nonaccidental cause. The loss function is

$$L = \begin{cases} 2Bv^T - \pi & J = 1 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 2 \quad 0 < T \le r - x \\ Bv^T - \pi & J = 1, 2 \quad T > r - x. \end{cases}$$

The equivalence principle requires that $\mathrm{E}[L] = 0$, or

$$\pi = B\left[\int_0^{r-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t)\, dt + \int_0^{\infty} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t)\, dt\right].$$

A measure of the dispersion due to the random natures of time and cause of death is provided by $\mathrm{Var}(L) = \mathrm{E}[L^2]$. One can verify that, for this case,

$$\mathrm{Var}(L) = B^2\left[3\int_0^{r-x} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(1)}(t)\, dt + \int_0^{\infty} v^{2t} \, {}_tp_x^{(\tau)} \, \mu_x^{(\tau)}(t)\, dt\right] - \pi^2.$$

In the general case, with actuarial present value given by (11.2.1), we have

$$\mathrm{Var}(L) = \mathrm{E}[L^2] = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t - \bar{A})^2 \, {}_tp_x^{(\tau)} \, \mu_x^{(j)}(t)\, dt,$$

which can be reduced to

$$\mathrm{Var}(L) = \sum_{j=1}^{m} \int_0^{\infty} (B_{x+t}^{(j)} \, v^t)^2 \, {}_tp_x^{(\tau)} \, \mu_{x(t)}^{(j)}\, dt - (\bar{A})^2. \tag{11.2.7}$$

11.3 Benefit Premiums and Reserves

We examine, in this section, a method of paying for benefits included in a life insurance policy in a multiple decrement setting. Often these extra benefits are included in life insurance contracts on a policy rider basis; that is, a specified extra premium is charged for the extra benefit, and a separate reserve is held for this benefit. The extra premium is payable only for as long as the benefit has value. In the case of double indemnity it is common to pay the extra amount only for death from accidental means before a specific age, such as 65, and thus the specified extra premiums would be payable only until that age.

We henceforth consider the double indemnity benefit as such a benefit. The model is not complete because the possibility of withdrawal, with a corresponding

withdrawal benefit, is not included. We consider withdrawal benefits in Section 11.4, but most of this subject is discussed in Chapters 15 and 16.

Consider the fully discrete model for a whole life policy to a person age 30 with a double indemnity rider. The benefit amount is one for nonaccidental death, decrement $J = 1$, and is two for death by accidental means, decrement $J = 2$. The principle of equivalence is now applied twice, once for the premium payable for life for the policy without the rider and once for the premium payable to age 65 for the extra benefit payable on accidental death before age 65.

For the policy without the rider the benefit level is one under either decrement 1 or 2 and the premium is payable for life. Thus

$$P_{30}^{(\tau)} = \frac{\sum_{k=0}^{\infty} v^{k+1}\, {}_kp_{30}^{(\tau)}\, q_{30+k}^{(\tau)}}{\sum_{k=0}^{\infty} v^k\, {}_kp_{30}^{(\tau)}}. \tag{11.3.1}$$

The benefit premium for the rider reflects that the premium is payable through age 64, and its benefit amount, payable under decrement 2 only, is unity. It is given by

$${}_{35}P_{30}^{(2)} = \frac{\sum_{k=0}^{34} v^{k+1}\, {}_kp_{30}^{(\tau)}\, q_{30+k}^{(2)}}{\sum_{k=0}^{34} v^k\, {}_kp_{30}^{(\tau)}}. \tag{11.3.2}$$

We now display the benefit reserve for the policy with the rider for years prior to attaining age 65:

$${}_kV = \sum_{h=0}^{\infty} v^{h+1}\, {}_hp_{30+k}^{(\tau)}\, q_{30+k+h}^{(\tau)} + \sum_{h=0}^{34-k} v^{h+1}\, {}_hp_{30+k}^{(\tau)}\, q_{30+k+h}^{(2)}$$

$$- \left(P_{30}^{(\tau)} \sum_{h=0}^{\infty} v^h\, {}_hp_{30+k}^{(\tau)} + {}_{35}P_{30}^{(2)} \sum_{h=0}^{34-k} v^h\, {}_hp_{30+k}^{(\tau)} \right).$$

This reserve is the sum of a reserve on the base policy plus a reserve on a policy that pays only on failure through decrement 2. The reserve on the base policy is a Chapter 7 benefit reserve for a fully discrete whole life insurance with $q_x = q_x^{(\tau)} = q_x^{(1)} + p_x^{(2)}$.

11.4 Withdrawal Benefit Patterns That Can Be Ignored in Evaluating Premiums and Reserves

A single decrement model for an individual life insurance benefit with annual premiums and reserves was built in Chapters 6 through 8. In that model the timing and, perhaps, the amount of benefit payments are determined by the time of death of the insured, and premiums are paid until death or the end of the premium period as specified in the policy. In practice, there is no way to prevent the cessation

of premium payments by the policyholder before death or the end of the premium period. In this situation an issue arises about how to reconcile the interests of the parties to the policy for which a model derived from multiple decrement theory is appropriate. Public policy considerations that should guide the reconciliation of the interests of the insurance system and the terminating insured have been subject to discussion since the early days of insurance.

Before premiums and reserves can be determined, a guiding principle must be adopted. A guiding principle is required as well in the determination of ***nonforfeiture benefits,*** those benefits that will not be lost because of the premature cessation of premium payments. In this section we adopt a simple operational principle, one that is, in effect, close to that adopted in U.S. insurance regulation. The principle is that the withdrawing insured receives a value such that the benefit, premium, and reserve structure, built using the single decrement model, remains appropriate in the multiple decrement context.

This principle is motivated by a particular concept of equity about the treatment of the two classes of policyholders, those who terminate before the basic insurance contract is fulfilled and those who continue. Clearly several concepts of what constitutes equity are possible, ranging from the view that terminating policyholders have not fulfilled the contract, and are therefore not entitled to nonforfeiture benefits, to the view that a terminating policyholder should be returned to his original position by the return of the accumulated value of all premiums, perhaps less an insurance charge. The concept of equity, which is the foundation of the principle adopted in the United States, is an intermediate one; that is, withdrawing life insurance policyholders are entitled to nonforfeiture benefits, but these benefits should not force a change in the price-benefit structure for continuing policyholders.

To illustrate some of the implications of this principle, we will develop a model for a whole life policy on a fully continuous payment basis with death and withdrawal benefits. The force of withdrawal is denoted by $\mu_x^{(2)}(t)$ with $\mu_x^{(\tau)}(t) = \mu_x^{(1)}(t) + \mu_x^{(2)}(t)$. For multiple decrement models, it is required that

$$\int_0^\infty \mu_{x+t}^{(\tau)}\, dt = \infty$$

so that

$$\lim_{t\to\infty} {}_tp_x^{(\tau)} = 0,$$

but it is not necessary for $\mu_x^{(2)}(t)$ and the derived ${}_tp_x'^{(2)}$ to have these properties.

We assume that the introduction of withdrawals into the model does not change the force of mortality, which is labeled for this development $\mu_x^{(1)}(t)$ in both the single and double decrement models. In other words, time-until-death and

time-until-withdrawal will be assumed to be independent, but this assumption may not be realized in practice. This issue was discussed in Chapter 10.

We start our model by specializing (8.6.4) to the case of a whole life insurance and single decrement premiums and reserves:

$$\frac{d}{dt}\,{}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)\,[1 - {}_t\bar{V}(\bar{A}_x)]. \tag{11.4.1}$$

Recalling from Section 10.2 that

$$\frac{d}{dt}\,{}_tp_x^{(\tau)} = -{}_tp_x^{(\tau)}[\mu_x^{(1)}(t) + \mu_x^{(2)}(t)],$$

we can express the following derivative as

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)] &= v^t\,{}_tp_x^{(\tau)}\{\bar{P}(\bar{A}_x) + \delta\,{}_t\bar{V}(\bar{A}_x) - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)]\}\\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)]\\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x) - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)].\end{aligned} \tag{11.4.2}$$

The progress of the reserves for a whole life insurance that includes withdrawal benefit ${}_tV(A_x)$ using premiums and reserves derived from a double decrement model is analogous to (11.4.1) and is shown in (11.4.3). In this expression, the superscript ^{2} denotes premiums and reserves based on the double decrement model:

$$\begin{aligned}\frac{d}{dt}\,[{}_t\bar{V}(\bar{A}_x)^2] &= \bar{P}(\bar{A}_x)^2 + \delta\,{}_t\bar{V}(\bar{A}_x)^2\\ &\quad - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^2] - \mu_x^{(2)}(t)[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^2].\end{aligned} \tag{11.4.3}$$

The last term in (11.4.3) is the net cost of withdrawal when the reserve ${}_tV(A_x)^2$ is treated as a savings fund available to offset benefits [see (8.4.5)]. Thus,

$$\begin{aligned}\frac{d}{dt}\,[v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^2] &= v^t\,{}_tp_x^{(\tau)}\,\{\bar{P}(\bar{A}_x)^2 + \delta\,{}_t\bar{V}(\bar{A}_x)^2 - \mu_x^{(1)}(t)[1 - {}_t\bar{V}(\bar{A}_x)^2]\\ &\quad - \mu_{x+t}^{(2)}[{}_t\bar{V}(\bar{A}_x) - {}_t\bar{V}(\bar{A}_x)^2]\}\\ &\quad - v^t\,{}_tp_x^{(\tau)}\,{}_t\bar{V}(\bar{A}_x)^2\,[\delta + \mu_x^{(1)}(t) + \mu_x^{(2)}(t)]\\ &= v^t\,{}_tp_x^{(\tau)}[\bar{P}(\bar{A}_x)^2 - \mu_x^{(1)}(t) - \mu_x^{(2)}(t)\,{}_t\bar{V}(\bar{A}_x)].\end{aligned} \tag{11.4.4}$$

Combining (11.4.2) and (11.4.4), we obtain

$$\frac{d}{dt}\,\{v^t\,{}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^2 - {}_t\bar{V}(\bar{A}_x)]\} = v^t\,{}_tp_x^{(\tau)}[\bar{P}(A_x)^2 - \bar{P}(\bar{A}_x)]. \tag{11.4.5}$$

We now integrate (11.4.5) from $t = 0$ to $t = \infty$ to obtain

$$0 = \bar{a}_x^{(\tau)}[\bar{P}(\bar{A}_x)^2 - \bar{P}(\bar{A}_x)], \tag{11.4.6}$$

which implies that

$$\bar{P}(\bar{A}_x)^{\underline{2}} = \bar{P}(\bar{A}_x).$$

Thus (11.4.5) reduces to

$$\frac{d}{dt}\{v^t\ {}_tp_x^{(\tau)}[{}_t\bar{V}(\bar{A}_x)^{\underline{2}} - {}_t\bar{V}(\bar{A}_x)]\} = 0,$$

which, with the initial condition that

$${}_0\bar{V}(\bar{A}_x)^{\underline{2}} = {}_0\bar{V}(\bar{A}_x),$$

implies that

$${}_t\bar{V}(\bar{A}_x)^{\underline{2}} = {}_t\bar{V}(\bar{A}_x) \qquad \text{for all } t \geq 0. \tag{11.4.7}$$

Therefore, if the withdrawal benefit in a double decrement model whole life insurance, fully continuous payment basis, is equal to the reserve under the single decrement model, the premium and reserves under the double decrement model are equal to the premium and reserves under the single decrement model. This result is not directly applied to the practical problem of defining nonforfeiture benefits. However, it does suggest the basic idea of how to minimize the impact of withdrawal or nonforfeiture benefits on premiums and reserves (determined under a single decrement model). These ideas are developed further in Chapter 16.

The ideas of this section are closely related to Example 6.6.2, where it was demonstrated that if the death benefit during the premium-paying period for a deferred life annuity is the accumulated value of the premiums, then the premium does not depend on the mortality assumption during the deferral period. This idea is elaborated in Example 11.4.1.

Example 11.4.1

A continuously paid life annuity issued on (x) provides an income benefit commencing at age $x + n$ at an annual rate of 1. The benefit for death (decrement $J = 1$) or withdrawal (decrement $J = 2$) during the n-year deferral period, paid at the moment of death, will be the accumulated benefit premiums with interest at the rate used in the premium calculation. Premiums are paid continuously from age x to $x + n$ or to the age of decrement, if less than $x + n$.

a. Formulate a loss variable.

b. Determine the annual benefit premium rate π using the principle of equivalence.

c. Determine the benefit reserve at time t, $0 \leq t \leq n$.

Solution:

a.
$$L = \begin{cases} \pi\, v^T\, \bar{s}_{\overline{T}|} - \pi\, \bar{a}_{\overline{T}|} & 0 \leq T \leq n, \quad J = 1, 2 \\ v^n\, \bar{a}_{\overline{T-n}|} - \pi\, \bar{a}_{\overline{n}|} & T > n, \qquad J = 1. \end{cases}$$

b. Applying the principle of equivalence, we obtain

$$\mathrm{E}[L] = \int_n^\infty (v^n\, \bar{a}_{\overline{t-n}|} - \pi\, \bar{a}_{\overline{n}|})\ {}_tp_x^{(\tau)}\ \mu_x^{(1)}(t)\ dt.$$

This yields

$$v^n \, {}_np_x^{(\tau)} \, \bar{a}_{x+n} = \pi \, \bar{a}_{\overline{n}|} \, {}_np_x^{(\tau)} \quad \text{and} \quad \pi = \frac{v^n \, \bar{a}_{x+n}}{\bar{a}_{\overline{n}|}} = \frac{\bar{a}_{x+n}}{\bar{s}_{\overline{n}|}}.$$

c. The reserve at time t, $t \leq n$, viewed prospectively, is given by

$$\begin{aligned}
&\int_0^{n-t} (\pi \, v^s \, \bar{s}_{\overline{t+s}|} - \pi \, \bar{a}_{\overline{s}|}) \, {}_sp_{x+t}^{(\tau)} \, \mu_x^{(\tau)}(t+s) \, ds \\
&\quad + \int_{n-t}^{\infty} (v^{n-t} \, \bar{a}_{\overline{s-(n-t)}|} - \pi \, \bar{a}_{\overline{n-t}|}) \, {}_sp_{x+t}^{(\tau)} \, \mu_x^{(1)}(t+s) \, ds \\
&= \pi \, \bar{s}_{\overline{t}|}(1 - {}_{n-t}p_{x+t}^{(\tau)}) + {}_{n-t|}\bar{a}_{x+t} - \pi \, \bar{a}_{\overline{n-t}|} \, {}_{n-t}p_{x+t}^{(\tau)} \\
&= \pi \, \bar{s}_{\overline{t}|}.
\end{aligned}$$

The simplification of the last term comes from the definition of π in part (b). The benefit premium and reserve during the deferred period can be viewed as derived from a zero decrement model. ▼

11.5 Valuation of Pension Plans

Two sets of assumptions are needed to determine the actuarial present values of pension plan benefits and of contributions to support these benefits. These sets can be identified as demographic (the service table and survival functions for retired lives, disabled lives, and perhaps lives who have withdrawn) and economic (investment return and salary scale) assumptions.

11.5.1 Demographic Assumptions

A starting point for the valuation of pension plan benefits is a multiple decrement (service) table constructed to represent a survivorship group of participants subject, in the various years of active service, to given probabilities of

- Withdrawal from service
- Death in service
- Retirement for disability, and
- Retirement for age-service.

The notations for these probabilities for the year of age x to $x + 1$ are $q_x^{(w)}$, $q_x^{(d)}$, $q_x^{(i)}$, and $q_x^{(r)}$, respectively. These are consistent with the notations developed in Chapter 10. Also, we use the survivorship function $l_x^{(\tau)}$ from Chapter 10, which satisfies

$$l_{x+1}^{(\tau)} = l_x^{(\tau)}[1 - (q_x^{(w)} + q_x^{(d)} + q_x^{(i)} + q_x^{(r)})] = l_x^{(\tau)} \, p_x^{(\tau)}.$$

This function can be used to evaluate such expression as ${}_kp_x^{(\tau)}$, thus,

$${}_kp_x^{(\tau)} = \frac{l_{x+k}^{(\tau)}}{l_x^{(\tau)}}.$$

One can also proceed by direct recursion, namely,

$$ {}_kp_x^{(\tau)} = {}_{k-1}p_x^{(\tau)}\, p_{x+k-1}^{(\tau)}. $$

The forces of decrement related to a service table will be continuous at most ages. They will be denoted by $\mu_x^{(w)}(t)$, $\mu_x^{(d)}(t)$, $\mu_x^{(i)}(t)$, and $\mu_x^{(r)}(t)$. At some ages, discontinuities may occur. This occurs most frequently at age α, the first eligible age for retirement. We generally assume that decrements are spread across each year of age.

In the early years of service, withdrawal rates tend to be high, and the benefit for withdrawal may be only the participant's contributions, if any, possibly accumulated with interest. After a period of time, for example, 5 years, withdrawal rates will be somewhat lower, and the withdrawing participant may be eligible for a deferred pension. If these conditions hold, it may be necessary to use select rates of withdrawal for an appropriate number of years. Conditions for disability retirement may also indicate a need for a select basis. The mathematical modifications to a select basis are relatively easy to make, and the theory is more adaptable if select functions are used. In this chapter we denote the age of entry by x, but we do not otherwise indicate whether an aggregate table, select table, or select-and-ultimate table is intended.

The Illustrative Service Table in Appendix 2B illustrates a service table for entry age 30, earliest age for retirement $\alpha = 60$, and with no probability of active service beyond age 71. Here $l_{71}^{(\tau)} = 0$.

As noted earlier, the principal benefits under a pension plan are annuities to eligible beneficiaries. For the valuation of such annuity benefits, it is necessary to adopt appropriate mortality tables that will differ if retirement is for disability, for age-service, or perhaps withdrawal. The corresponding annuity values will be indicated by post-fixed superscripts. The continuous annuity value is used as a convenient means of approximating the actual form of pension payment that usually is monthly, but may have particular conditions as to initial and final payments.

11.5.2 Projecting Benefit Payment and Contribution Rates

A common form of pension plan is one that defines the rate of retirement income by formula. These plans are called ***defined benefit plans.*** Some pension plans define benefit income rates as a function of the level of compensation at or near retirement. In these cases, it is necessary to estimate future salaries to value the benefits. Sponsor contributions are also often expressed as a percentage of salary, so here too estimation of future salaries is important. To accomplish these estimations, we define the following salary functions:

$(AS)_{x+h}$ is the actual annual salary rate at age $x + h$, for a participant who entered at age x and is now at attained age $x + h$,

$(ES)_{x+h+t}$ is the projected (estimated) annual salary rate at age $x + h + t$.

Further, we assume that we have a salary scale function S_y to use for these projections, such that

$$(ES)_{x+h+t} = (AS)_{x+h} \frac{S_{x+h+t}}{S_{x+h}}. \tag{11.5.1}$$

The salary functions S_y may reflect merit and seniority increases in salary as well as those caused by inflation. For example, in the Illustrative Service Table, $S_y = (1.06)^{y-30} s_y$, where the s_y factor represents the progression of salary due to individual merit and experience increases, and the 6% accumulation factor is to allow for long-term effects of inflation and of increases in productivity of all members of the plan. As was the case of the $l_x^{(\tau)}$ function, one of the values of S_y can be chosen arbitrarily. For instance, in the Illustrative Service Table, S_{30} is taken as unity. The S_y function is usually assumed to be a step function, with constant level throughout any given year of age.

We now move to the problem of estimating the benefit level for a pension plan. For this purpose, we introduce the function $R(x, h, t)$ to denote the projected annual income benefit rate to commence at age $x + h + t$ for a participant, who entered h years ago at age x. Both x and h are assumed to be integers. We assume that the income benefit rate remains level during payout so that when we come to expressing the actuarial present value of the benefit at time of retirement, it will simply be $R(x, h, t)\bar{a}^r_{x+h+t}$. As stated in the previous section, the post-fixed superscript r indicates that a mortality table appropriate for retired lives should be used.

We now consider several common types of income benefit rate functions $R(x, h, t)$. The estimation procedure falls into two groups. First, there are functions that do not depend on salary levels. For other types of benefit formulas, which depend on future salaries, the projected annual income rate must be estimated. There are those that depend on either the final salary rate or on an average salary rate over the last several years prior to retirement. There are also formulas that depend on the average salary over the career with the plan sponsor. The following are examples of the more common types of benefit formula together with their estimation.

a. Consider an income benefit rate that is a fraction d of the final salary rate. Thus $R(x, h, t) = d(ES)_{x+h+t}$. Here we estimate the final salary from the current salary at age $x + h$ by $(ES)_{x+h+t} = (AS)_{x+h}(S_{x+h+t} / S_{x+h})$ so that $R(x, h, t) = d(AS)_{x+h}(S_{x+h+t} / S_{x+h})$.
b. A ***final m-year average salary benefit*** rate is a fraction d of the average salary rate over the last m years prior to retirement. We illustrate this in the common case where $m = 5$. In this case, if $t > 5$, an estimate of the average salary over the last 5 years is given as

$$(AS)_{x+h} \frac{0.5\, S_{x+h+k-5} + S_{x+h+k-4} + S_{x+h+k-3} + S_{x+h+k-2} + S_{x+h+k-1} + 0.5\, S_{x+h+k}}{5\, S_{x+h}}$$

where k is the greatest integer in t.

The thinking behind this expression is that if retirement occurs at midyear, the current year's salary is earned only for the last half year of service. A notation in common usage for the above average is ${}_5Z_{x+h+k} / S_{x+h}$. If the participant

is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.

The above formulas do not reflect the amount of service of the participant at retirement. We now look at three formulas where the benefits are proportional to the number of years of service at retirement.

c. Consider an income benefit that is d times the total number of years of service, including any fraction in the final year of employment. In this case $R(x, h, t) = d(h + t)$. If only whole years of service are to be counted, then $R(x, h, t) = d(h + k)$, where k is the greatest integer in t.

d. Consider an income benefit rate that is the product of a fraction d of the final 5-year average salary and the number of years of service at retirement. A typical formula would be, where d is a designated fraction,

$$= d(h + t)(AS)_{x+h} \frac{{}_5Z_{x+h+k}}{S_{x+h}}.$$

Again, if the participant is within 5 years of possible retirement, account could be taken of actual rather than projected salaries.

e. Consider an income benefit rate that is d times the number of years of service times the average salary over the entire career. Such a benefit formula is called a ***career average benefit.*** This formula is equivalent to a benefit rate of a fraction d of the entire career earnings of the retiree.

The analysis of career average retirement benefits breaks naturally into two parts, one for past service for which the salary information is known and one for future service where salaries must be estimated. Here past salaries enter into the valuation of benefits for all participants and not just for those participants very near to retirement age. If the total of past salaries for a participant at age $x + h$ is denoted by $(TPS)_{x+h}$, the benefit rate attributed to past service is $d(TPS)_{x+h}$. The retirement income benefit rate based on future service is given by

$$d(AS)_{x+h} \frac{S_{x+h} + S_{x+h+1} + \cdots + S_{x+h+k-1} + 0.5\, S_{x+h+k}}{S_{x+h}},$$

where k is the greatest integer in t and retirements are assumed to occur at midyear.

Finally, we display one benefit formula where the service component for participants with a large number of years of service at retirement is modified.

f. Consider an income benefit rate that is the product of the 3-year final average salary and 0.02 times the number of years of service at retirement for the first 30 years of service with an additional 0.01 per year of service above 30 years. Following (d),

$$= 0.02\ (h + t)\ (AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} \qquad h + t \leq 30$$

$$= [0.30 + 0.01\ (h + t)]\ (AS)_{x+h} \frac{{}_3Z_{x+h+k}}{S_{x+h}} \qquad h + t > 30.$$

11.5.3 Defined-Benefit Plans

We now seek to develop formulas for actuarial present values of such benefits and of the contributions expected to be used to fund the promised benefits. We do so first for a general case of a defined-benefit plan and then examine a specific example that includes a typical pattern of defined benefits.

Let us first look at the evaluation of, and approximation to, the actuarial present value for an age-retirement benefit. Assume that the benefit rate function has been found as $R(x, h, t)$ and that the benefit involves life annuities with no certain period. We can then write an integral expression for the actuarial present value of the retirement benefit as

$$\text{APV} = \int_{\alpha-x-h}^{\infty} v^t \, {}_tp_{x+h}^{(\tau)} \, \mu_x^{(r)}(h + t) \, R(x, h, t) \, \bar{a}_{x+h+t}^{r} \, dt. \tag{11.5.2}$$

As in Section 11.2, we approximate the integral for practical calculation of the actuarial present value. To do so, we write

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \int_0^1 v^s \, {}_sp_{x+h+k}^{(\tau)} \, \mu_x^{(r)}(h + k + s) \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^{r} \, ds.$$

By assuming a uniform distribution of retirements in each year of age, we can rewrite this as

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^k \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \int_0^1 v^s \, R(x, h, k + s) \, \bar{a}_{x+h+k+s}^{r} \, ds.$$

Using the midpoint approximation for the remaining integrals gives

$$\text{APV} = \sum_{k=\alpha-x-h}^{\infty} v^{k+1/2} \, {}_kp_{x+h}^{(\tau)} \, q_{x+h+k}^{(r)} \, R(x, h, k + 1/2) \, \bar{a}_{x+h+k+1/2}^{r}. \tag{11.5.3}$$

Formula (11.5.3) is the general means by which we calculate the actuarial present value of retirement and, by extension, other benefits of a pension plan.

We now present an example that shows the types of calculations that might be used for the valuation of the several benefits of a hypothetical defined-benefit pension plan.

Example 11.5.1

Find the actuarial present values of the following benefits for a participant who was hired 3 years ago at age 30 and who currently has a salary of $45,000.

a. Retirement income for any participant of at least age 65 or whenever the sum of the attained age and the number of years of service exceeds a total of 90. The benefit is in the form of a 10-year certain and life annuity, payable monthly, at an annual rate of 0.02 times the final 5-year average salary times the total number of years of service, including any final fraction.

b. Retirement income for any participant with at least 5 years of service upon withdrawal. The benefit and income benefit rate formula is as for age retirement. However, the initial payment of the annuity is deferred until the earliest possible date of age retirement had the participant continued in the active status.
c. Retirement income for those disabled participants too young for age retirement. The income benefit rate is the larger of 50% or the percentage based on years of service, uses the average salary over the preceding 5 years, and is for a 10-year certain and life annuity.
d. Lump sum benefit for those participants who die while still in active status. The benefit amount is two times the salary rate at the time of death.

Solution:

The participant was hired at age 30 and so is eligible for retirement at age 60 [60 + (60 − 30) = 90]. Assuming midyear retirements, the income benefit rate is given by

$$= 0.02\,(h + t)\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\,(3 + k + 0.5)\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

for retirements starting in the year following age $33 + k$. The actuarial present value of age-retirement benefits is approximated by

$$\text{APV} = 900 \sum_{k=27}^{\infty} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(r)} \times (3.5 + k)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,\bar{a}^{r}_{\overline{33+k+1/2:\overline{10}|}}.$$

Benefits are paid for those withdrawing from active status between ages 35 and 60. After attaining age 60, the withdrawals are classified as retirements. The income benefit rate is again given by

$$= 0.02\,(h + t)\,(AS)_{x+h}\,\frac{{}_5Z_{x+h+k}}{S_{x+h}}$$

$$= 0.02\,(3 + k + 0.5)\,(45{,}000)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}.$$

For withdrawals at ages 35 through 37, adjustments using actual wage data rather than the Z function could be made. The actuarial present value of the withdrawal benefits is approximated by

$$\text{APV} = 900 \sum_{k=2}^{26} v^{k+1/2}\,{}_kp_{30+3}^{(\tau)}\,q_{30+3+k}^{(w)} \times (3.5 + k)\,\frac{{}_5Z_{30+3+k}}{S_{30+3}}\,{}_{27-k-1/2|}\,\bar{a}^{w}_{\overline{33+k+1/2:\overline{10}|}}.$$

For the disability benefit the income benefit rate function makes a distinction between disabilities starting before and after the participant has worked for 25 years. Thus the income benefit rate is

$$= 0.5\ (AS)_{x+h}\ \frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.5\ (45{,}000)\ \frac{{}_5Z_{30+3+k}}{S_{30+3}} \qquad \text{for } 0 \le k \le 21$$

$$= 0.02\ (h + t)\ (AS)_{x+h}\ \frac{{}_5Z_{x+h+k}}{S_{x+h}} = 0.02\ (3 + k + 0.5)\ (45{,}000)\ \frac{{}_5Z_{30+3+k}}{S_{30+3}}$$

$$\text{for } 22 \le k \le 26.$$

Thus, the actuarial present value of the disability benefits is approximated by

$$\text{APV} = 22{,}500 \sum_{k=0}^{21} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(i)}\ \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ \bar{a}_{\overline{33+k+1/2:\overline{10}|}}^{\,i}$$

$$+ 900 \sum_{k=22}^{26} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(i)}\ \frac{{}_5Z_{30+3+k}}{S_{30+3}}\ (3.5 + k)\ \bar{a}_{\overline{33+k+1/2:\overline{10}|}}^{\,i}.$$

For the death benefit, the projected lump sum benefit amount is

$$= 2\ (AS)_{x+h}\ \frac{S_{x+h+k}}{S_{x+h}}$$

$$= 2\ (45{,}000)\ \frac{S_{30+3+k}}{S_{30+3}}.$$

Thus, the actuarial present value of the death benefits is approximated by

$$\text{APV} = 90{,}000 \sum_{k=0}^{\infty} v^{k+1/2}\ {}_kp_{30+3}^{(\tau)}\ q_{30+3+k}^{(d)}\ \frac{S_{30+3+k}}{S_{30+3}}.$$

▼

There are many funding or budgeting methods available to assure that contributions are made to the plan in an orderly and appropriate manner. An overview of these methods is presented in Chapter 20.

11.5.4 Defined-Contribution Plans

The principal benefit under a pension plan is normally the deferred annuity for age-service retirement. In ***defined-contribution plans,*** the actuarial present value is simply the accumulation under interest of contributions made by or for the participant, and the benefit is an annuity that can be purchased by such accumulation. The accumulated amount is typically available upon death and, under certain conditions, upon withdrawal before retirement. We examine the interplay between the rate of contribution and the rate of income provided to the participant in the following example. The defined-contribution rate can be determined with a retirement income goal. The risk that the goal will not be achieved is held by the participant of the plan, not the sponsor. Budget constraints on the sponsor may, of course, restrict the amount of the contributions.

Example 11.5.2

Find the contribution level for the sponsor to provide for age-retirement at age 65 that has as its objective a 10-year certain and life annuity with an initial benefit rate of 50% of the average salary over the 5 years between ages 60 and 65. The contribution rate, which is to be applied as a proportion of salary, is calculated for a new participant at age 30. Assume that there are no withdrawal benefits for the first 5 years but after that the accumulated contributions are vested, that is, become the property of the withdrawing participant, and will be applied toward an annuity to start no earlier than age 60. (An active participant who becomes disabled is treated as a withdrawal and is covered by a separate disability income coverage for the period between the date of disability and age 65 at which time a regular age retirement commences.) Upon death after the end of the 5-year vesting period but before retirement income has commenced, the accumulated contributions are paid out.

Solution:

During the first 5 years our model contains both mortality and withdrawal decrements. The scheme of benefits in this plan after the 5-year vesting period are similar to those discussed in Example 11.4.1. Thus for the period after time 5 until the participant attains age 65, we use a zero decrement model.

Let us start by calculating the actuarial present value of a contribution rate of c times the annual salary rate, assumed for convenience to be 1, at age 30:

$$\text{APV} = c\left[\sum_{k=0}^{4} v^{k+1/2}\,\frac{S_{30+k}}{S_{30}}\,{}_kp_{30}^{(\tau)}\left(1-\frac{1}{2}\,q_{30+k}^{(\tau)}\right)\right.$$
$$\left. + {}_5p_{30}^{(\tau)}\sum_{k=5}^{34} v^{k+1/2}\,\frac{S_{30+k}}{S_{30}}\right]. \qquad (11.5.4)$$

In (11.5.4) contributions are assumed to occur at midyear, and the projected salary rate at age $30 + k$ is (S_{30+k} / S_{30}) times 1, the initial salary at age 30. For $k = 0$ to 4, we need survivorship to time $k + (1 / 2)$. Since the model has no decrements after time 5, survivorship to time 5 is all that is needed for $k = 5$ to 34.

We now estimate the desired benefit payment rate at age 65 as the first step in estimating the actuarial present value of the target benefits. The average salary projected to be earned between the ages of 60 and 65 is $(S_{60} + S_{61} + S_{62} + S_{63} + S_{64}) / (5\ S_{30})$. The desired benefit rate is one-half of this, and the actuarial present value of the target benefit is given by

$$\text{APV} = {}_5p_{30}^{(\tau)}\,v^{35}\,(0.5)\,\frac{S_{60}+S_{61}+S_{62}+S_{63}+S_{64}}{5\ S_{30}}\,\bar{a}^{r}_{\overline{65:\overline{10}|}}. \qquad (11.5.5)$$

In this expression, survivorship is required only to duration 5, but we must discount for interest for the full 35 years.

We now equate the two actuarial present values to solve for c, the sponsor's contribution rate, which will be applied to all future salary payment in accordance with the plan to achieve the stated retirement income goal. ▼

Some plans of this type have both sponsor and participant contributions. It is common here for some kind of matching between the size of the sponsor contribution and the size of the participant contribution.

11.6 Disability Benefits with Individual Life Insurance

In Section 11.5.3 we discuss disability benefits included in pension plans. We now turn to disability benefits commonly found with individual life insurance. Provision can be made for the waiver of life insurance premiums during periods of disability. Alternatively, policies can contain a provision for a monthly income, sometimes related to the face amount, if disability occurs. The multiple decrement model is appropriate for studying these provisions.

The usual disability clause provides a benefit for total disability. Total disability can require a disability severe enough to prevent engaging in any gainful occupation, or it can require only the inability to engage in one's own occupation. Total disability that has been continuous for a period of time specified in the policy, called the ***waiting or elimination period,*** qualifies the policyholder to receive benefit payments. The waiting period can be 1, 3, 6, or 12 months. In policies with waiver of premium, it is common to make the benefits ***retroactive,*** that is, to refund any premiums paid by the insured during the waiting periods. Coverage is only for disabilities that occur prior to a disability benefit expiry age, typically 60 or 65. However, benefits in the form of an annuity, either as disability income or as waiver or premiums, will often continue to a higher age, typically the maturity date or paid-up date of the life insurance policy.

11.6.1 Disability Income Benefits

Let us start by expressing the actuarial present value of a disability income benefit of 1,000 per month issued to (x) under coverage expiring at age y and with income running to age u. We assume that the waiting period is m months. Using notation from Chapter 10 and earlier sections of Chapter 11, we can express the actuarial present value as a definite integral as

$$\bar{A} = \int_0^{y-x} v^t \, {}_tp_x^{(\tau)} \, \mu_x^{(i)}(t) \, v^{m/12} \, {}_{m/12}p_{[x+t]}^{i} \, (12{,}000 \, \ddot{a}_{[x+t]:\overline{u-x-t-m/12}|}^{(12)i}) \, dt. \qquad (11.6.1)$$

The i superscript on ${}_{m/12}p_{[x+t]}^{i}$ indicates a survival probability for a disabled life. We now break up the integral into separate integrals for each year. Upon making the assumption of uniform distribution for the disability decrement within each year of age and replacing t by $k + s$, we obtain an expression for the actuarial present value much like (11.2.5):

$$\bar{A} = 12{,}000 \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \, q_{x+k}^{(i)} \, v^{m/12}$$

$$\times \int_0^1 v^s \, {}_{m/12}p_{[x+k+s]}^{i} \, \ddot{a}_{[x+k+s]+m/12:\,\overline{u-x-k-s-m/12}|}^{(12)i} \, ds. \qquad (11.6.2)$$

A simplification of this formula occurs when the decrement i (disability) is defined to occur only if the person who is disabled survives to the end of the waiting period of m months. If death occurs during the waiting period, the decrement is regarded as death. This means that the disabled life survivorship factor, ${}_{m/12}p_{[x+k+s]}^{i}$, is unnecessary as it has been taken into account in the definition of $q_{x+k}^{(i)}$. We note that it also means that the attained age at entry into the disabled life state is reached at the completion of the waiting period and is so indicated in the select age of the disabled life annuity function.

By the midpoint method the integrals in (11.6.2) are evaluated as

$$\int_0^1 v^s \, \ddot{a}_{[x+k+s+m/12]:\overline{u-x-k-s-m/12}|}^{(12)i} \, ds = v^{1/2} \, \ddot{a}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|}^{(12)i}. \qquad (11.6.3)$$

With these two changes (11.6.2) can be written as

$$A = 12{,}000 \sum_{k=0}^{y-x-1} v^{k+1/2} \, {}_kp_x^{(\tau)} q_{x+k}^{(i)} v^{m/12} \ddot{a}_{[x+k+1/2+m/12]:\overline{u-x-k-1/2-m/12}|}^{(12)i}. \qquad (11.6.4)$$

11.6.2 Waiver-of-Premium Benefits

Let us go through the same process for a waiver of premium benefit. We assume that the premium, P, to be waived is payable g times per year for life. The primary difference between this and the disability income benefit is that the waiver benefit is a g-thly payment annuity starting at the first premium due date after the end of the waiting period.

We start with a special case with its actuarial present value written with the definite integrals already broken down to individual years of time of disablement. The case chosen is the waiver of semiannual premiums, payable for life, in the event of a disability occurring prior to age y and continuing through a 4-month waiting period. We further assume that the benefits are retroactive by which we mean that for a premium paid to the insurer on a due date during the waiting period, reimbursement with interest at the valuation rate will be made. This will increase the number of integrals within each year of age because disabilities that start within the first 2 months of each half year do not have premiums due during the waiting period:

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \left[\int_0^{1/6} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1/2-s|}\ddot{a}_{[x+k+s]}^{(2)i} \, ds \right.$$
$$+ \int_{1/6}^{1/2} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{1-s|}\ddot{a}_{[x+k+s]}^{(2)i} + {}_{4/12}p_{[x+k+s]}^{i} \, v^{1/2-s} \frac{1}{2} \right) ds$$
$$+ \int_{1/2}^{2/3} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \, {}_{1-s|}\ddot{a}_{[x+k+s]}^{(2)i} \, ds$$
$$\left. + \int_{2/3}^{1} v^s \, {}_sp_{x+k}^{(\tau)} \, \mu_x^{(i)}(k+s) \left({}_{3/2-s|}\ddot{a}_{[x+k+s]}^{(2)i} + {}_{4/12}p_{[x+k+s]}^{i} \, v^{1-s} \frac{1}{2} \right) ds \right]. \quad (11.6.5)$$

If we incorporate the assumption of a uniform distribution of disability within each year of age and include only those disabilities that survive the waiting period as disabilities, (11.6.5) becomes

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} q_{x+k}^{(i)} \left[\int_0^{1/6} v^{s+4/12} \, {}_{1/2-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} \, ds \right.$$
$$+ \int_{1/6}^{1/2} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} + v^{1/2} \frac{1}{2} \, ds$$
$$+ \int_{1/2}^{2/3} v^{s+4/12} \, {}_{1-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} \, ds$$
$$\left. + \int_{2/3}^{1} v^{s+4/12} \, {}_{3/2-s-4/12|}\ddot{a}_{[x+k+s+4/12]}^{(2)i} + v \frac{1}{2} \, ds \right]. \quad (11.6.6)$$

We now use the midpoint approximate integration method for each of the several integrals within each year of age to obtain

$$\bar{A} = P \sum_{k=0}^{y-x-1} v^k \, {}_kp_x^{(\tau)} \, q_{x+k}^{(i)} \left[\frac{1}{6} v^{5/12} \, {}_{1/12|}\ddot{a}_{[x+k+5/12]}^{(2)i} \right.$$
$$+ \frac{1}{3} \left(v^{2/3} \, {}_{1/3|}\ddot{a}_{[x+k+2/3]}^{(2)i} + v^{1/2} \frac{1}{2} \right) + \frac{1}{6} v^{11/12} \, {}_{1/12|}\ddot{a}_{[x+k+11/12]}^{(2)i}$$
$$\left. + \frac{1}{3} \left(v^{7/6} \, {}_{1/3|}\ddot{a}_{[x+k+7/6]}^{(2)i} + v \frac{1}{2} \right) \right]. \quad (11.6.7)$$

11.6.3 Benefit Premiums and Reserves

Equivalence principle benefit premiums for disability income and waiver of premium benefits are found by equating the actuarial present value of benefits to the actuarial present value of premiums. For the waiver benefit discussed in Section 11.6.2, the annual benefit premium, ${}_{y-x}\Pi_x$, equals $\bar{A}$ of (11.6.7) divided by $\ddot{a}_{x:\overline{y-x}|}^{(\tau)}$.

Active life benefit reserves, that is, the reserve when premiums are not being waived, are most conveniently expressed by a premium difference formula:

$${}_kV = ({}_{y-x-k}\Pi_{x+k} - {}_{y-x}\Pi_x) \, \ddot{a}_{x+k:\overline{y-x-k}|}^{(\tau)}. \quad (11.6.8)$$

The terminal reserve for a disabled life is the actuarial present value of future disability benefits, calculated on the assumption that the insured has incurred a

disability. The amount of premium waived, or disability income rate, is multiplied by the actuarial present value of an appropriate disabled life annuity. This value takes into account the age at disablement, the duration since disablement, and the terminal age for benefits.

11.7 Notes and References

We have not defined insurer's losses and studied their variances in this chapter. Formula (11.2.7) gave a means of doing so if we consider the total benefits for all causes of decrements. If we consider only a single benefit, such as the retirement benefit, there is more than one way of defining losses. The usual concept is that premiums and reserves, for a benefit in regard to a particular cause of decrement, apply only to that decrement. Thus, if decrement due to a second cause occurs, then, with respect to the first cause, there is zero benefit and a gain emerges. An insurer's loss based on this concept would lead, for example, to (11.4.3). However, losses defined in this way may have nonzero covariances, so that the loss variance for all benefits is not the sum of the loss variances for the individual benefits.

Alternatively, one may consider that when a particular cause of decrement occurs, the reserves accumulated for the benefits in regard to all the other causes are released to offset the benefit outgo for the given cause. In this case, the loss random variables defined for the benefits for the several causes of decrement have zero-valued covariances, and the loss variance for all benefits is the sum of the loss variances for the individual benefits. However, the premiums and reserves for the individual benefits are more difficult to compute on this second basis and individually differ significantly from those on the usual basis. For insights into these matters, see Hickman (1964).

The result stated in Section 11.4 concerning the neutral impact on premiums and reserves when a withdrawal benefit equals the reserve on the death benefit does not hold for fully discrete insurances. This was pointed out by Nesbitt (1964), who reported on work by Schuette. The problem results from the fact that, in the discrete model, the probability of withdrawal

$$q_{x+k}^{(w)} = \int_0^1 \exp\left[-\int_0^t \mu_x^{(\tau)}(k+s)\,ds\right] \mu_x^{(w)}(k+t)\,dt$$

depends on the force of mortality.

While there are many papers and a number of books dealing with pension fund mathematics, it seems useful for the purposes of this introductory treatment to refer only to other actuarial texts with similar chapters; see, for example, Hooker and Longley-Cooke (1957), Jordan (1967), and Neill (1977). These authors stress the formulation of actuarial present values in terms of pension commutation functions and the use of tables of such functions to carry out computations.

In contrast, a major portion of our presentation has been in terms of integrals and approximating sums, with the integrands or summands expressed in terms of

basic functions. These approximating sums can be computed by various processes that may or may not make use of commutation functions. For pension benefits determined by complex eligibility or income conditions, it can be more flexible and efficient to calculate by processes not requiring extensive formulation by commutation functions. An opposing view, indicating the power of commutation functions for expressing actuarial present values and controlling their computation, is given by Chamberlain (1982).

There is an alternative foundation for constructing a model for disability insurance. In Section 11.6 we used a multiple decrement model that did not explicitly provide for recovery from disability. Models with several states of disability, with provision for transition from state to state, have been developed. These models are frequently the foundation of long-term care insurance. Hoem (1988) provides an introduction with valuable references to these ideas.

The multiple decrement model developed in Chapter 10 and applied in this chapter can be viewed as being made up of $m + 1$ states; m are called absorbing states in that it is impossible to return from them to the active state. These m states are associated with the m causes of decrement, and the remaining state is associated with continuing survival. If some of the m decrements are not absorbing, but are such that transition to the active state or one of the other nonabsorbing states is possible, a more complex but possibly more realistic model results. Estimation of the probabilities of transition among the states can be difficult because the probabilities can depend on the path followed to the current state.

Exercises

Section 11.2

11.1. Employees enter a benefit plan at age 30. If an employee remains in service until retirement, the employee receives an annual pension of 300 times years of service. If the employee dies in service before retirement, the beneficiary is paid 20,000 immediately. If the employee withdraws before age 70 for any reason except death, the member receives a deferred (to age 70) life annuity of 300 times years of service. Give an expression, in terms of integrals and continuous annuities, for the actuarial present value of these benefits at age 30.

11.2. Let $J = 1$ represent death by accidental means and $J = 2$ represent death by other means. You are given that

(i) $\delta = 0.05$

(ii) $\mu_x^{(1)}(t) = 0.005$ for $t \geq 0$ where $\mu_x^{(1)}(t)$ is the force of decrement for death by accidental means.

(iii) $\mu_x^{(2)}(t) = 0.020$ for $t \geq 0$.

A 20-year term insurance policy, payable at the moment of death, is issued to a life age (x) providing a benefit of 2 if death is by accidental means and

providing a benefit of 1 for other deaths. Find the expectation and variance of the present value of benefits random variable.

Section 11.4

11.3. A double decrement model is defined by $\mu_x^{(1)}(t) = \mu_x^{(2)}(t) = 1/(a - t)$, $0 \leq t < a$.

a. In the single decrement model with decrement (1) only, the prospective loss variable at duration t is given by

$$_tL^{\underline{1}} = v^{T(x)-t} \qquad 0 < t \leq T(x),\ J = 1.$$

Confirm that

$$_tV^{\underline{1}} = \frac{1 - e^{-\delta(a-t)}}{\delta(a - t)} = A^{\underline{1}}_{x+t}.$$

b. In the double decrement model, the prospective loss variable at duration t is given by

$$\begin{aligned} _tL^{\underline{2}} &= v^{T(x)-t} && 0 < t \leq T(x),\ J = 1 \\ &= v^{T(x)-t}\ {}_{T(x)}V^{\underline{1}} && 0 < t \leq T(x),\ J = 2. \end{aligned}$$

Confirm that $\mathrm{E}[_tL^{\underline{2}} | T \geq t] = {}_tV^{\underline{1}}$.

Section 11.5

11.4. A pension plan valuation assumes a linear salary scale function satisfying $S_{20} = 1$. If $(ES)_{45} = 2(AS)_{25}$, find S_x for $x \geq 20$.

11.5. A new pension plan with two participants, (35) and (40), provides annual income at retirement equal to 2% of salary at the final rate times the number of years of service, including any fraction of a year. If

(i) Salary increases occur continuously,

(ii) For (40), $(AS)_{40} = 50{,}000$ and $S_{40+t} = 1 + 0.06t$, and

(iii) For (35), $(AS)_{35} = 35{,}000$ and $S_{35+t} = 1 + 0.10t$,

calculate the maximum value of $[R(40, 0, t) - R(35, 0, t)]$ for $t \geq 0$.

11.6. It is assumed that, for a new participant entering at age 30, there will be annual increases in salary at the rate of 5% per year to take care of the effects of inflation and increases in productivity. In addition, it is assumed that promotion raises of 10% of the existing salary will occur at ages 40, 50, and 60.

a. Construct a salary scale function, S_{30+k}, to express these assumptions.

b. Write an expression for the actuarial present value of contributions of 10% of future salary for a new entrant with annual salary 24,000 and with increases in salary according to the scale constructed in (a).

11.7. Every year, a plan sponsor contributes 10% of that portion of each participant's salary in excess of a certain amount. That amount is 15,000 this year and will increase by 5% annually. Express the actuarial present value of the

sponsor's contribution for a participant entering now at age 35 with a salary of 40,000.

11.8. A plan provides for an income benefit rate, payable from retirement to age 65, of 2% of the final 3-year average salary for each year of service. After age 65 the income benefit rate is 1-1/3% of the final 3-year average salary for each year of service.

a. For a participant age 50, who entered service at age 30 and currently has a salary of 48,000, express the actuarial present value of the participant's benefit if the earliest retirement age is 55 and there is no mandatory retirement age.

b. If the maximum number of years to be credited in the plan is 35, express the actuarial present value of the benefit for the above participant.

c. Give an expression for the actuarial present value of the income benefit associated with past service for the above participant.

11.9. A career average plan provides a retirement income of 2% of aggregate salary during a participant's years of service. The earliest age of retirement is 58, and all retirements are completed by age 68. For a participant age 50 who entered service at age 30 and has 450,000 total of past salaries with a current salary of 42,000, write expressions for

a. The participant's total income benefit rate in case of retirement at exact age 65

b. The participant's midyear total income benefit rate in case of retirement between ages 65 and 66

c. The actuarial present value of this participant's retirement benefit for past service

d. The actuarial present value of this participant's retirement benefit for future service.

11.10. A new participant in a pension plan, age 45, has a choice of two benefit options:

(1) A defined-contribution plan with contributions of 20% of salary each year. Contributions are made at the beginning of each year and earn 5% per year. Accumulated contributions are used to purchase a monthly life annuity-due.

(2) A defined-benefit plan with an annual benefit, payable monthly, of 40% of the final 2-year average salary.

You are given that (a) $\ddot{a}_{65}^{(12)} = 10$ and (b) $S_{45+k} = (1.05)^k$ for $k \geq 0$ where S_y is a step function, constant over each year of age. Assuming that retirement occurs at exact age 65 and that the participant survives to retirement, calculate the ratio of the expected monthly payment under the defined-contribution plan to that under the defined-benefit plan.

11.11. Display definite integrals for the actuarial present values of the following possible benefits of an employee benefit package. Assume that the employee is currently age 40 and earning 40,000 annually. This employee was hired at

age 25 and has received a total of 320,000 in salary since hire. Retirement benefits are available only after age 55, and withdrawal benefits are only available before age 55 in the form of an annuity with payments deferred until the employee reaches age 55.

a. A retirement benefit at the annual benefit rate of 50% of the final salary.
b. A retirement benefit at the rate of 0.015 times the product of the final salary rate multiplied by the exact number of years (including fractions) of service at the moment of retirement.
c. A withdrawal benefit using the benefit income formula in (b).
d. A retirement benefit at the rate of 0.025 times the total salary paid over the whole career to the employee.
e. A withdrawal benefit using the benefit income formula in (d).

11.12. A retirement benefit consisting of a continuous annuity, payable for life, is part of an employer's benefit package. The annual benefit income rate is 60% of the salary rate applicable at the moment of retirement for retirements between ages 60 and 70. For retirements after attaining age 70, the benefit rate is 60% of the salary rate applicable between ages 69 and 70. Give an approximating sum for the actuarial present value of this benefit for a person age 30 who has just been hired at a salary of 35,000.

Section 11.6

11.13. a. Give an expression for the annual benefit premium, payable to age 60, for a disability income insurance issued to (35) of 2,000 per month payable to age 65 in case (35) becomes disabled before age 60 and survives a waiting period of 6 months.
b. Give an expression for the active life benefit reserve at the end of 10 years for the insurance in (a).

Miscellaneous

11.14. The Hattendorf theorem for the fully continuous model as stated in Exercise 8.24 can be restated in the definitions and notation of this chapter for the fully continuous multiple decrement model:

$$\text{Var}({}_0L^{\underline{m}}) = \sum_{j=1}^{m} \int_0^{\infty} [v^t\,(B^{(j)}_{x+t} - {}_t\bar{V}^{\underline{m}})]^2\; {}_tp_x^{(\tau)}\; \mu_x^{(j)}(t)\; dt.$$

Confirm that this result holds for the fully continuous whole life insurance discussed in Section 11.4.

Outline of solution:

a. Confirm that the loss random variable for this insurance is

$${}_0L^{\underline{2}} = \begin{cases} v^T - \bar{P}(\bar{A}_x)^{\underline{2}}\; \bar{a}_{\overline{T}|} & 0 \le T,\ J = 1 \\ v^T\; {}_T\bar{V}(\bar{A}_x) - \bar{P}(\bar{A}_x)^{\underline{2}}\; \bar{a}_{\overline{T}|} & 0 \le T,\ J = 2. \end{cases}$$

b. Use (11.4.6) and (11.4.7) to rewrite the differential equation (11.4.3) and then employ the integrating factor $e^{-\delta t}$ to obtain the solution

$$v^t \, {}_t\bar{V}(\bar{A}_x) = \bar{P}(\bar{A}_x)\, \bar{a}_{\overline{t}|} - \int_0^t e^{-\delta s}\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds.$$

c. Use the result of part (b) to modify both lines of the definition of ${}_0L^{\underline{2}}$ in part (a) and then show that

$$\begin{aligned}\text{Var}({}_0L^{\underline{2}}) = \int_0^\infty \Bigg\{ & v^t [1 - {}_t\bar{V}(\bar{A}_x)] \\ & - \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \Bigg\}^2 {}_tp_x^{(\tau)} \mu_x^{(1)}(t)\, dt \\ & + \int_0^\infty \Bigg\{ \int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds \Bigg\}^2 {}_tp_x^{(\tau)}\, \mu_x^{(2)}(t)\, dt.\end{aligned}$$

d. Perform the indicated squaring operation on the factor in the integrand of the first integral in part (c) and combine the two integrals that include $\{\int_0^t v^s\, \mu_x^{(1)}(s)\, [1 - {}_s\bar{V}(\bar{A}_x)]\, ds\}^2$ as a component of the integrand. Then use integration by parts to obtain

$$\text{Var}({}_0L^{\underline{2}}) = \int_0^\infty \{v^t\, [1 - {}_t\bar{V}(\bar{A}_x)]\}^2\, {}_tp_x^{(\tau)}\, \mu_x^{(1)}(t)\, dt.$$

This result provides a reduction of variance argument for establishing the withdrawal benefit as ${}_t\bar{V}(\bar{A}_x)$.



COLLECTIVE RISK MODELS FOR A SINGLE PERIOD

12.1 Introduction

In Chapters 3 through 11 we considered models for long-term insurances. The inclusion of interest in these models was essential. In this chapter we return to a topic introduced in Chapter 2, namely, short-term insurance policies. Consequently interest will be ignored. The purpose of this chapter is to present an alternative to the individual policy model discussed in Chapter 2.

The individual risk model of Chapter 2 considers individual policies and the claims produced by each policy. Then aggregate claims are obtained by summing over all the policies in the portfolio.

For the ***collective risk model*** we assume a random process that generates claims for a portfolio of policies. This process is characterized in terms of the portfolio as a whole rather than in terms of the individual policies comprising the portfolio. The mathematical formulation is as follows: Let N denote the number of claims produced by a portfolio of policies in a given time period. Let X_1 denote the amount of the first claim, X_2 the amount of the second claim, and so on. Then,

$$S = X_1 + X_2 + \cdots + X_N \tag{12.1.1}$$

represents the aggregate claims generated by the portfolio for the period under study. The number of claims, N, is a random variable and is associated with the frequency of claim. The individual claim amounts $X_1, X_2, \ldots$ are also random variables and are said to measure the severity of claims.

In order to make the model tractable, we usually make two fundamental assumptions:

1. $X_1, X_2, \ldots$ are identically distributed random variables.
2. The random variables $N, X_1, X_2, \ldots$ are mutually independent.

Expression (12.1.1) will be called a random sum, and unless stated otherwise, assumptions (1) and (2) will be made concerning its components.

A principal tool for developing the theory of this chapter is the moment generating function (m.g.f.). These functions provide a simple but powerful means for the reader to gain a working knowledge of the collective theory of risk. A reader who has not worked with them recently would do well to review the m.g.f.'s, means, and variances of the widely used probability distributions summarized in Appendix 5.

12.2 The Distribution of Aggregate Claims

In this section we see how the distribution of aggregate claims in a fixed time period can be obtained from the distribution of the number of claims and the distribution of individual claim amounts.

Let $P(x)$ denote the common d.f. of the independent and identically distributed X_i's. Let X be a random variable with this d.f. Then let

$$p_k = \mathrm{E}[X^k] \tag{12.2.1}$$

denote the k-th moment about the origin, and

$$M_X(t) = \mathrm{E}[e^{tX}] \tag{12.2.2}$$

denote the m.g.f. of X. In addition, let

$$M_N(t) = \mathrm{E}[e^{tN}] \tag{12.2.3}$$

denote the m.g.f. of the number of claims, and let

$$M_S(t) = \mathrm{E}[e^{tS}] \tag{12.2.4}$$

denote the m.g.f. of aggregate claims. The d.f. of aggregate claims will be denoted by $F_S(s)$.

Using (2.2.10) and (2.2.11), in conjunction with assumptions (1) and (2) of Section 12.1, we obtain

$$\mathrm{E}[S] = \mathrm{E}[\mathrm{E}[S|N]] = \mathrm{E}[p_1 N] = p_1 \mathrm{E}[N] \tag{12.2.5}$$

and

$$\begin{aligned} \mathrm{Var}(S) &= \mathrm{E}[\mathrm{Var}(S|N)] + \mathrm{Var}(\mathrm{E}[S|N]) \\ &= \mathrm{E}[N \mathrm{Var}(X)] + \mathrm{Var}(p_1 N) \\ &= \mathrm{E}[N] \mathrm{Var}(X) + p_1^2 \mathrm{Var}(N) \end{aligned} \tag{12.2.6}$$

where $\mathrm{Var}(X) = p_2 - p_1^2$.

The result stated in (12.2.5), that the expected value of aggregate claims is the product of the expected individual claim amount and the expected number of claims, is not surprising. Expression (12.2.6) for the variance of aggregate claims also has a natural interpretation. The variance of aggregate claims is the sum of two components where the first is attributed to the variability of individual claim amounts and the second to the variability of the number of claims.

In a similar fashion we derive an expression for the m.g.f. of S:

$$\begin{aligned} M_S(t) &= \mathrm{E}[e^{tS}] = \mathrm{E}[\mathrm{E}[e^{tS}|N]] \\ &= \mathrm{E}[M_X(t)^N] = \mathrm{E}[e^{N\log M_X(t)}] \\ &= M_N[\log M_X(t)]. \end{aligned} \tag{12.2.7}$$

Example 12.2.1

Assume that N has a geometric distribution; that is, the p.f. of N is given by

$$\Pr(N = n) = pq^n \qquad n = 0, 1, 2, \ldots \tag{12.2.8}$$

where $0 < q < 1$ and $p = 1 - q$. Determine $M_S(t)$ in terms of $M_X(t)$.

Solution:

Since

$$M_N(t) = \mathrm{E}[e^{tN}] = \sum_{n=0}^{\infty} p(qe^t)^n = \frac{p}{1 - qe^t},$$

(12.2.7) tells us that

$$M_S(t) = \frac{p}{1 - qM_X(t)}. \tag{12.2.9}$$

▼

To derive the d.f. of S, we distinguish according to how many claims occur and use the law of total probability,

$$\begin{aligned} F_S(x) &= \Pr(S \le x) = \sum_{n=0}^{\infty} \Pr(S \le x|N = n) \Pr(N = n) \\ &= \sum_{n=0}^{\infty} \Pr(X_1 + X_2 + \cdots + X_n \le x) \Pr(N = n). \end{aligned} \tag{12.2.10}$$

In terms of the convolution defined in Section 2.3, we can write

$$\begin{aligned} \Pr(X_1 + X_2 + \cdots + X_n \le x) &= P{*}P{*}P{*}\cdots{*}P(x) \\ &= P^{*n}(x), \end{aligned} \tag{12.2.11}$$

which is the n-th convolution of P defined in Chapter 2. Recall that

$$P^{*0}(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}$$

Thus (12.2.10) becomes

$$F_S(x) = \sum_{n=0}^{\infty} P^{*n}(x) \Pr(N = n). \tag{12.2.12}$$

If the individual claim amount distribution is discrete with p.f. $p(x) = \Pr(X = x)$, the distribution of aggregate claims is also discrete. By analogy with the above derivation, the p.f. of S can be obtained directly as

$$f_S(x) = \sum_{n=0}^{\infty} p^{*n}(x) \Pr(N = n) \tag{12.2.13}$$

where

$$p^{*n}(x) = p * p * \cdots * p(x) = \Pr(X_1 + X_2 + \cdots + X_n = x) \tag{12.2.11A}$$

$$\text{and } p^{*0}(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Here the inequality sign in the probability symbol in (12.2.11) has been replaced by the equal sign.

Example 12.2.2

Consider an insurance portfolio that will produce zero, one, two, or three claims in a fixed time period with probabilities 0.1, 0.3, 0.4, and 0.2, respectively. An individual claim will be of amount 1, 2, or 3 with probabilities 0.5, 0.4, and 0.1, respectively. Calculate the p.f. and d.f. of the aggregate claims.

Solution:

The calculations are summarized below. Only nonzero entries are exhibited.

(1) x	(2) $p^{*0}(x)$	(3) $p^{*1}(x) = p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $f_S(x)$	(7) $F_S(x)$
0	1.0	—	—	—	0.1000	0.1000
1	—	0.5	—	—	0.1500	0.2500
2	—	0.4	0.25	—	0.2200	0.4700
3	—	0.1	0.40	0.125	0.2150	0.6850
4	—	—	0.26	0.300	0.1640	0.8490
5	—	—	0.08	0.315	0.0950	0.9440
6	—	—	0.01	0.184	0.0408	0.9848
7	—	—	—	0.063	0.0126	0.9974
8	—	—	—	0.012	0.0024	0.9998
9	—	—	—	0.001	0.0002	1.0000
n	0	1	2	3	—	—
$\Pr(N = n)$	0.1	0.3	0.4	0.2	—	—

Since there are at most three claims and each produces a claim amount of at most 3, we can limit the calculations to $x = 0, 1, 2, \ldots, 9$.

Column (2) lists the p.f. of a degenerate distribution with all the probability mass at 0. Column (3) lists the p.f. of the individual claim amount random variable. Columns (4) and (5) are obtained recursively by applying

$$\begin{aligned} p^{*(n+1)}(x) &= \Pr(X_1 + X_2 + \cdots + X_{n+1} = x) \\ &= \sum_y \Pr(X_{n+1} = y) \Pr(X_1 + X_2 + \cdots + X_n = x - y) \\ &= \sum_y p(y)\, p^{*n}(x - y). \end{aligned} \tag{12.2.14}$$

Since only three different claim amounts are possible, the evaluation of (12.2.14) will involve a sum of three or fewer terms. Next, (12.2.13) is used to compute the p.f. displayed in column (6). For this step, it is convenient to record the p.f. of N in the last row of the results. Finally, the elements of column (7) are obtained as partial sums of column (6). An alternative approach, not illustrated here, would be to perform the convolutions in terms of the d.f.'s, obtain $F_S(x)$ from (12.2.12), and calculate $f_S(x) = F_S(x) - F_S(x-1)$. ▼

If the claim amount distribution is continuous, it cannot be concluded that the distribution of S is continuous. If $\Pr(N = 0) > 0$, the distribution of S will be of the mixed type; that is, it will have a mass of probability at 0 and be continuous elsewhere. This idea is illustrated in the following example.

Example 12.2.3

In Example 12.2.1, add the assumption that

$$P(x) = 1 - e^{-x} \qquad x > 0;$$

that is, the individual claim amount distribution is exponential with mean 1. Then show that

$$M_S(t) = p + q\frac{p}{p-t} \tag{12.2.15}$$

and interpret the formula.

Solution:

First, we rewrite (12.2.9) as follows:

$$M_S(t) = p + q\frac{pM_X(t)}{1 - qM_X(t)}.$$

Then we substitute

$$M_X(t) = \int_0^\infty e^{tx}\, e^{-x}\, dx = (1-t)^{-1}$$

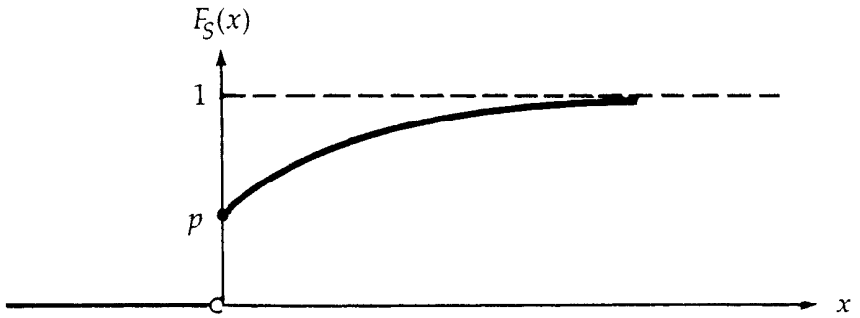
to obtain (12.2.15).

Since 1 is the m.g.f. of the constant 0 and $p/(p-t)$ is the m.g.f. of the exponential distribution with d.f. $1 - e^{-px}$, $x > 0$, (12.2.15) can be interpreted as a weighted average (with weights p and q, respectively). It follows that the d.f. of S is the corresponding weighted average of distributions. Thus, for $x > 0$

$$F_S(x) = p(1) + q(1 - e^{-px}) = 1 - qe^{-px}. \tag{12.2.16}$$

This distribution is of the mixed type. Its d.f. is shown in Figure 12.2.1. ▼

Graph of $F_S(x)$



12.3 Selection of Basic Distributions

In this section we discuss some issues in selecting the distribution of the number of claims N and the common distribution of the X_i's. As different considerations apply to these two selections, a separate subsection will be devoted to each.

12.3.1 The Distribution of *N*

One choice for the distribution of N is the Poisson with p.f. given by

$$\Pr(N = n) = \frac{\lambda^n e^{-\lambda}}{n!} \qquad n = 0, 1, 2, \ldots \tag{12.3.1}$$

where $\lambda > 0$. For the Poisson distribution, $\mathrm{E}[N] = \mathrm{Var}(N) = \lambda$. With this choice for the distribution of N, the distribution of S is called a ***compound Poisson distribution.*** Using (12.2.5) and (12.2.6), we have that

$$\mathrm{E}[S] = \lambda p_1 \tag{12.3.2}$$

and

$$\mathrm{Var}(S) = \lambda p_2 \,. \tag{12.3.3}$$

Substituting the m.g.f. of the Poisson distribution

$$M_N(t) = e^{\lambda(e^t - 1)} \tag{12.3.4}$$

into (12.2.7), we obtain the m.g.f. of the compound Poisson distribution,

$$M_S(t) = e^{\lambda[M_X(t) - 1]} \,. \tag{12.3.5}$$

The compound Poisson distribution has many attractive features, some of which are discussed in Section 12.4.

When the variance of the number of claims exceeds its mean, the Poisson distribution is not appropriate. In this situation, use of the negative binomial distribution has been suggested. The negative binomial distribution has a p.f. given by

$$\Pr(N = n) = \binom{r + n - 1}{n} p^r q^n \qquad n = 0, 1, 2, \ldots . \tag{12.3.6}$$

This distribution has two parameters: $r > 0$ and $0 < p < 1$; $q = 1 - p$. For this distribution, we have

$$M_N(t) = \left(\frac{p}{1 - q e^t}\right)^r, \tag{12.3.7}$$

$$\mathrm{E}[N] = \frac{rq}{p}, \tag{12.3.8}$$

and

$$\mathrm{Var}(N) = \frac{rq}{p^2}. \tag{12.3.9}$$

When a negative binomial distribution is chosen for N, the distribution of S is called a ***compound negative binomial distribution.*** Substituting from (12.3.8) and (12.3.9) into (12.2.5) and (12.2.6), we have

$$\mathrm{E}[S] = \frac{rq}{p} p_1 \tag{12.3.10}$$

and

$$\mathrm{Var}(S) = \frac{rq}{p} p_2 + \frac{rq^2}{p^2} p_1^2 . \tag{12.3.11}$$

Substituting from (12.3.7) into (12.2.7), we obtain

$$M_S(t) = \left[\frac{p}{1 - qM_X(t)}\right]^r. \tag{12.3.12}$$

We observe that the family of geometric distributions used in Examples 12.2.1 and 12.2.3 is contained as a special case ($r = 1$) of the two-parameter family of negative binomial distributions.

A family of distributions for the number of claims can be generated by assuming that the Poisson parameter Λ is a random variable with p.d.f. $u(\lambda)$, $\lambda > 0$, and that the conditional distribution of N, given $\Lambda = \lambda$, is Poisson with parameter λ. There are several situations in which this might be a useful way to consider the distribution of N. For example, consider a population of insureds where various classes of insureds within the population generate numbers of claims according to Poisson distributions with different values of λ for the various classes. If the relative frequency of the values of λ is denoted by $u(\lambda)$, we can use the law of total probability to obtain

$$\begin{aligned}\Pr(N = n) &= \int_0^\infty \Pr(N = n|\Lambda = \lambda)\, u(\lambda) d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} u(\lambda) d\lambda. \end{aligned} \tag{12.3.13}$$

Furthermore, using (2.2.10) and (2.2.11), we have

$$\mathrm{E}[N] = \mathrm{E}[\mathrm{E}[N|\Lambda]] = \mathrm{E}[\Lambda] \tag{12.3.14}$$

and

$$\begin{aligned}\mathrm{Var}(N) &= \mathrm{E}[\mathrm{Var}(N|\Lambda)] + \mathrm{Var}(\mathrm{E}[N|\Lambda]) \\ &= \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda).\end{aligned} \tag{12.3.15}$$

Also,

$$M_N(t) = \mathrm{E}[e^{tN}] = \mathrm{E}[\mathrm{E}[e^{tN}|\Lambda]] = \mathrm{E}[e^{\Lambda(e^t-1)}] = M_\Lambda(e^t - 1). \tag{12.3.16}$$

The equality,

$$\mathrm{E}[e^{tN}|\Lambda] = e^{\Lambda(e^t-1)},$$

follows from the hypothesis that the conditional distribution of N, given Λ, is Poisson with parameter Λ.

A comparison of (12.3.14) and (12.3.15) shows that, as in the case of the negative binomial distribution, $\mathrm{E}[N] < \mathrm{Var}(N)$. In fact, the negative binomial distribution can be derived in this fashion, which will be shown in the following example.

Example 12.3.1

Assume that $u(\lambda)$ is the gamma p.d.f. with parameters α and β,

$$u(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \qquad \lambda > 0 \tag{12.3.17}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy.$$

a. Show that the marginal distribution of N is negative binomial with parameters

$$r = \alpha, \qquad p = \frac{\beta}{1+\beta}. \tag{12.3.18}$$

b. By substituting $\mathrm{E}[\Lambda] = \alpha/\beta$ and $\mathrm{Var}(\Lambda) = \alpha/\beta^2$ into (12.3.14) and (12.3.15), verify (12.3.8) and (12.3.9).

Solution:

a. Substituting

$$M_\Lambda(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha \tag{12.3.19}$$

into (12.3.16), we have

$$\begin{aligned}M_N(t) = M_\Lambda(e^t - 1) &= \left[\frac{\beta}{\beta - (e^t - 1)}\right]^\alpha \\ &= \left\{\frac{\beta/(\beta+1)}{1 - [1 - \beta/(\beta+1)]e^t}\right\}^\alpha.\end{aligned} \tag{12.3.20}$$

Comparison of (12.3 20) with (12.3.7) confirms that this distribution for N is negative binomial with parameters $r = \alpha$,

$$p = \frac{\beta}{1+\beta}, \qquad (12.3.21)$$

$$q = 1 - p = \frac{1}{1+\beta}.$$

b. The suggested substitutions into (12.3.14) and (12.3.15) yield

$$\mathrm{E}[N] = \frac{\alpha}{\beta} = \frac{rq}{p}$$

as in (12.3.8) and

$$\mathrm{Var}(N) = \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{rq}{p}\left(1 + \frac{q}{p}\right) = \frac{rq}{p^2}$$

as in (12.3.9). ▼

The following is another example of a distribution for N that is obtained by mixing Poisson distributions.

Example 12.3.2

Assume that $u(\lambda)$ is the inverse Gaussian p.d.f. with parameters α and β. Exhibit the moment generating function of N, $\mathrm{E}[N]$, and $\mathrm{Var}(N)$.

Solution:

Example 2.3.5 contains the basic facts about the inverse Gaussian distribution.

Applying (12.3.16) yields

$$M_N(t) = M_\Lambda(e^t - 1) = e^{\alpha[1-[1-2(e^t-1)/\beta]^{1/2}]},$$

and from (12.3.14) and (12.3.15) we obtain

$$\mathrm{E}[N] = \mathrm{E}[\Lambda] = \frac{\alpha}{\beta}$$

and

$$\mathrm{Var}(N) = \mathrm{E}[\Lambda] + \mathrm{Var}(\Lambda)$$

$$= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} = \frac{\alpha(\beta+1)}{\beta^2}.$$

This distribution is called the ***Poisson inverse Gaussian distribution.*** ▼

Table 12.3.1 summarizes pertinent information on the compound distributions resulting from the selections for N discussed here.

Compound Distributions of S

$$S = \sum_{j=1}^{N} X_j$$

$N, X_1, X_2, \ldots$ are independent random variables. Each X_j has d.f. $P(x)$, m.g.f. $M_X(t)$, and $p_k = \mathrm{E}[X^k] \quad k = 1, 2, \ldots$

$$P^{*0}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$P^{*n}(x) = \begin{cases} \sum_{j=0}^{x} p(x-j)P^{*(n-1)}(j), \text{ or} \\ \int_0^x p(x-y)P^{*(n-1)}(y)dy \end{cases}$$

Definitions	Distribution Function, $F_S(x)$	Restrictions on Parameters	Moment Generating Function, $M_S(t)$	Mean	Variance
General	$\sum_{n=0}^{\infty} \Pr(N = n)P^{*n}(x)$	—	$M_N[\log M_X(t)]$	$p_1 \mathrm{E}[N]$	$\mathrm{E}[N](p_2 - p_1^2) + p_1^2 \mathrm{Var}(N)$
Compound Poisson	$\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} P^{*n}(x)$	$\lambda > 0$	$e^{\lambda[M_X(t)-1]}$	λp_1	λp_2
Compound Negative Binomial	$\sum_{n=0}^{\infty} \binom{r+n-1}{n} p^r q^n P^{*n}(x)$	$0 < p < 1$ $q = 1 - p$ $r > 0$	$\left[\frac{p}{1 - qM_X(t)}\right]^r qM_X(t) < 1$	$\frac{rqp_1}{p}$	$\frac{rqp_2}{p} + \frac{rq^2p_1^2}{p^2}$
Compound Poisson Inverse Gaussian	no known closed form	$\alpha > 0$ $\beta > 0$	$\exp\left\{-\alpha\left[1 - \left\{1 - \frac{2[M_X(t) - 1]}{\beta}\right\}^{1/2}\right]\right\}$	$\frac{\alpha}{\beta} p_1$	$\frac{\alpha}{\beta}\left(p_2 + \frac{p_1^2}{\beta}\right)$

12.3.2 The Individual Claim Amount Distribution

On the basis of (12.2.12) we see that convolutions of the individual claim amount distribution may be required. Thus, when possible, it is convenient to select that distribution from a family of distributions for which convolutions can be calculated easily either by formula or numerically. For example, if the claim amount has the normal distribution with mean μ and variance σ^2, then its n-th convolution is the normal distribution with mean $n\mu$ and variance $n\sigma^2$. For many types of insurance, the claim amount random variable is only positive, and its distribution is skewed to the right. For these insurances we might choose a gamma distribution that also has these properties. The n-th convolution of a gamma distribution with parameters α and β is also a gamma distribution but with parameters $n\alpha$ and β. This can be confirmed by noting from (12.3.19) that $M_X(t) = [\beta / (\beta - t)]^{\alpha}$, and hence the m.g.f. associated with $P^{*n}(x)$ is

$$M_X(t)^n = \left(\frac{\beta}{\beta - t}\right)^{n\alpha} \qquad t < \beta. \tag{12.3.22}$$

If the claim amounts have an exponential distribution with parameter 1, the p.d.f. is given by

$$p(x) = e^{-x} \qquad x > 0.$$

This is a gamma distribution with $\alpha = \beta = 1$. Then, by using (12.3.19), we conclude that the n-th convolution is a gamma distribution with parameters $\alpha = n$, $\beta = 1$; that is,

$$p^{*n}(x) = \frac{x^{n-1}\, e^{-x}}{(n-1)!} \qquad x > 0. \tag{12.3.23}$$

To obtain an expression for $P^{*n}(x)$, we perform integration by parts n times as follows:

$$\begin{aligned} 1 - P^{*n}(x) &= \int_x^{\infty} \frac{y^{n-1}\, e^{-y}}{(n-1)!}\, dy \\ &= -\frac{y^{n-1}}{(n-1)!}\, e^{-y}\Big|_x^{\infty} + \int_x^{\infty} \frac{y^{n-2}\, e^{-y}}{(n-2)!}\, dy \\ &= \frac{x^{n-1}}{(n-1)!}\, e^{-y} + [1 - P^{*(n-1)}(x)] \\ &= e^{-y} \sum_{i=0}^{n-1} \frac{x^i}{i!}. \end{aligned} \tag{12.3.24}$$

Then, using (12.2.12), we have

$$1 - F_S(x) = \sum_{n=1}^{\infty} \Pr(N = n) e^{-y} \sum_{i=0}^{n-1} \frac{x^i}{i!} \qquad x > 0. \tag{12.3.25}$$

This exponential distribution case shows that even with simple assumed distributions, the distribution of aggregate claims may not have a simple form. Therefore, it may be more practical to select a discrete claim amount distribution and calculate

the required convolutions numerically. For compound Poisson distributions it has been established that the convolution method can be shortened or, alternatively, that it can be bypassed by use of a recursive formula for directly calculating the distribution function of S. These computational shortcuts are discussed in the following section.

12.4 Properties of Certain Compound Distributions

In this section we discuss some mathematical properties of certain compound distributions. Two theorems concerning the compound Poisson are presented.

The first shows that the sum of independent random variables, each having a compound Poisson distribution, also has a compound Poisson distribution.

Theorem 12.4.1

If $S_1, S_2, \ldots, S_m$ are mutually independent random variables such that S_i has a compound Poisson distribution with parameter λ_i and d.f. of claim amount $P_i(x)$, $i = 1, 2, \ldots, m$, then $S = S_1 + S_2 + \cdots + S_m$ has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i \tag{12.4.1}$$

and

$$P(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} P_i(x) \tag{12.4.2}$$

Proof:

We let $M_i(t)$ denote the m.g.f. of $P_i(x)$. According to (12.3.5), the m.g.f. of S_i is

$$M_{S_i}(t) = \exp\{\lambda_i[M_i(t) - 1]\}.$$

By the assumed independence of $S_1, \ldots, S_m$, the m.g.f. of their sum is

$$M_S(t) = \prod_{i=1}^{m} M_{S_i}(t) = \exp\left\{\sum_{i=1}^{m} \lambda_i[M_i(t) - 1]\right\}.$$

Finally, we rewrite the exponent to obtain

$$M_S(t) = \exp\left\{\lambda\left[\sum_{i=1}^{m} \frac{\lambda_i}{\lambda} M_i(t) - 1\right]\right\}. \tag{12.4.3}$$

Since this is the m.g.f. of the compound Poisson distribution, specified by (12.4.1) and (12.4.2), the theorem follows. ■

This result has two important consequences for building insurance models. First, if we combine m insurance portfolios, where the aggregate claims of each of the portfolios have compound Poisson distributions and are mutually independent,

then the aggregate claims for the combined portfolio will also have a compound Poisson distribution. Second, we can consider a single insurance portfolio for a period of m years. Here we assume independence among the annual aggregate claims for the m years and that the aggregate claims for each year have a compound Poisson distribution. It is not necessary that the annual aggregate claims distributions be identical. Then it follows from Theorem 12.4.1 that the total claims for the m-year period will have a compound Poisson distribution.

Example 12.4.1

Let $x_1, x_2, \ldots, x_m$ be m different numbers and suppose that $N_1, N_2, \ldots, N_m$ are mutually independent random variables. Further, suppose that N_i $(i = 1, 2, \ldots, m)$ has a Poisson distribution with parameter λ_i. What is the distribution of

$$x_1N_1 + x_2N_2 + \cdots + x_mN_m? \tag{12.4.4}$$

Solution:

By interpreting x_iN_i to have a compound Poisson distribution with Poisson parameter λ_i and a degenerate claim amount distribution at x_i, we can apply Theorem 12.4.1 to establish that the sum in (12.4.4) has a compound Poisson distribution with

$$\lambda = \sum_{i=1}^{m} \lambda_i$$

and p.f. of claim amount $p(x)$ where

$$p(x) = \begin{cases} \dfrac{\lambda_i}{\lambda} & x = x_i, \quad i = 1, 2, \ldots, m \\ 0 & \text{elsewhere.} \end{cases} \tag{12.4.5}$$

▼

We show in Theorem 12.4.2 that the construction in Example 12.4.1 is reversible: that is, every compound Poisson distribution with a discrete claim amount distribution can be represented as a sum of the form (12.4.4). We let $x_1, x_2, \ldots, x_m$ denote the discrete values for individual claim amounts and let

$$\pi_i = p(x_i) \qquad i = 1, 2, \ldots, m \tag{12.4.6}$$

denote their respective probabilities. Let N_i be the number of terms in (12.1.1) that are equal to x_i. Then, by collecting terms, we see that

$$S = x_1N_1 + x_2N_2 + \cdots + x_mN_m\,. \tag{12.4.7}$$

In general, the N_i's of (12.4.7) are dependent random variables. However, in the special case of a compound Poisson distribution for S, they are independent, as is shown in Theorem 12.4.2.

Before stating Theorem 12.4.2, we cite some properties of the ***multinomial distribution*** that are used in the proof. For the multinomial, each of n independent

trials result in one of m different outcomes. The probability that a trial ends in outcome i is denoted by π_i. We denote the random variable that counts the number of outcomes i in n trials by N_i. Then

$$1 = \sum_{i=1}^{m} \pi_i, \qquad n = \sum_{i=1}^{m} N_i,$$

and the joint p.f. of $N_1, N_2, \ldots, N_m$ is given by

$$\Pr(N_1 = n_1, N_2 = n_2, \ldots, N_m = n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!} \pi_1^{n_1} \pi_2^{n_2} \cdots \pi_m^{n_m}. \quad (12.4.8)$$

By using (12.4.8) we obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = (\pi_1 e^{t_1} + \pi_2 e^{t_2} + \cdots + \pi_m e^{t_m})^n. \quad (12.4.9)$$

The multivariate discrete distribution with p.f. given by (12.4.8) and m.g.f. given by (12.4.9) is called a multinomial distribution with parameters $n, \pi_1, \ldots, \pi_m$.

Theorem 12.4.2

If S, as given in (12.4.7), has a compound Poisson distribution with parameter λ and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6), then

a. $N_1, N_2, \ldots, N_m$ are mutually independent.

b. N_i has a Poisson distribution with parameter $\lambda_i = \lambda \pi_i$, $i = 1, 2, \ldots, m$.

Proof:

We start by defining the m.g.f. of the joint distribution of $N_1, N_2, \ldots, N_m$ by use of (2.2.10) for conditional expectations. Note that for a fixed number of independent claims (trials) where each claim results in one of m claim amounts, the numbers of claims of each amount have a multinomial distribution with parameters $n, \pi_1, \pi_2, \ldots, \pi_m$. Hence, given

$$N = \sum_{i=1}^{m} N_i = n,$$

the conditional distribution of $N_1, N_2, \ldots, N_m$ is this multinomial distribution. For this case, we use (12.4.9) to obtain

$$\begin{aligned} \mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] &= \sum_{n=0}^{\infty} \mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right) \middle| N = n\right] \Pr(N = n) \\ &= \sum_{n=0}^{\infty} (\pi_1 e^{t_1} + \cdots + \pi_m e^{t_m})^n \frac{e^{-\lambda} \lambda^n}{n!}. \end{aligned} \quad (12.4.10)$$

We now perform the required summation by recognizing (12.4.10) as a Taylor series expansion of an exponential function. We obtain

$$\mathrm{E}\left[\exp\left(\sum_{i=1}^{m} t_i N_i\right)\right] = \exp(-\lambda)\exp\left(\lambda \sum_{i=1}^{m} \pi_i e^{t_i}\right)$$

$$= \prod_{i=1}^{m} \exp[\lambda \pi_i (e^{t_i} - 1)]. \qquad (12.4.11)$$

Since this is the product of m functions each of a single variable t_i, (12.4.11) shows the mutual independence of the N_i's. Furthermore, if we set $t_i = t$, and $t_j = 0$ for $j \neq i$ in (12.4.11), we obtain

$$\mathrm{E}[\exp(tN_i)] = \exp[\lambda \pi_i (e^t - 1)], \qquad (12.4.12)$$

which is the m.g.f. of the Poisson distribution with parameter $\lambda \pi_i$. This proves statement (b). ■

Formula (12.4.7) and Theorem 12.4.2 provide an alternative method for tabulating a compound Poisson distribution with a discrete claim amount distribution. First, we compute the p.f.'s of $x_1 N_1$, $x_2 N_2$, . . . , $x_m N_m$. Since the nonzero entries for the p.f. of $x_i N_i$ are at multiples of x_i and are Poisson probabilities, this is an easy task. Then the convolution of these m distributions is calculated to obtain the p.f. of S. This method is particularly convenient if m, the number of different claim amounts, is small. Even if a continuous distribution has been selected for the individual claim amounts, a discrete approximation can sometimes be used with this alternative method to produce a satisfactory approximation to the distribution of S. The basic and the alternative methods for tabulating the distribution of S are compared in the following example.

Example 12.4.2

Suppose that S has a compound Poisson distribution with $\lambda = 0.8$ and individual claim amounts that are 1, 2, or 3 with probabilities 0.25, 0.375, and 0.375, respectively. Compute $f_S(x) = \Pr(S = x)$ for $x = 0, 1, \ldots, 6$.

Solution:

For the basic method, the calculations parallel those in Example 12.2.2 and are summarized below.

Basic Method Calculations

(1) x	(2) $p^{*0}(x)$	(3) $p(x)$	(4) $p^{*2}(x)$	(5) $p^{*3}(x)$	(6) $p^{*4}(x)$	(7) $p^{*5}(x)$	(8) $p^{*6}(x)$	(9) $f_S(x)$
0	1	—	—	—	—	—	—	0.449329
1	—	0.250000	—	—	—	—	—	0.089866
2	—	0.375000	0.062500	—	—	—	—	0.143785
3	—	0.375000	0.187500	0.015625	—	—	—	0.162358
4	—	—	0.328125	0.070313	0.003906	—	—	0.049905
5	—	—	0.281250	0.175781	0.023438	0.000977	—	0.047360
6	—	—	0.140625	0.263672	0.076172	0.007324	0.000244	0.030923
n	0	1	2	3	4	5	6	
$e^{-0.8}\frac{(0.8)^n}{n!}$	0.449329	0.359463	0.143785	0.038343	0.007669	0.001227	0.000164	

For the alternative method outlined in this section, the calculations are displayed below.

Alternative Method Calculations

(1)	(2)	(3)	(4)	(5)	(6)
(x)	$\Pr(1N_1 = x)$	$\Pr(2N_2 = x)$	$\Pr(3N_3 = x)$	$\Pr(N_1 + 2N_2 = x)$ $= (2)*(3)$	$\Pr(N_1 + 2N_2 + 3N_3 = x)$ $= (4)*(5) = f_S(x)$
0	0.818731	0.740818	0.740818	0.606531	0.449329
1	0.163746	—	—	0.121306	0.089866
2	0.016375	0.222245	—	0.194090	0.143785
3	0.001092	—	0.222245	0.037201	0.162358
4	0.000055	0.033337	—	0.030973	0.049905
5	0.000002	—	—	0.005703	0.047360
6	0.000000	0.003334	0.033337	0.003287	0.030923
i	1	2	3		
λ_i	0.2	0.3	0.3		
	$\frac{e^{-0.2}(0.2)^x}{x!}$	$\frac{e^{-0.3}(0.3)^{x/2}}{(x/2)!}$	$\frac{e^{-0.3}(0.3)^{x/3}}{(x/3)!}$		

For the application of the formulas of this section, we note that $m = 3$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $\lambda_1 = \lambda\, p(1) = 0.2$, $\lambda_2 = \lambda\, p(2) = 0.3$, and $\lambda_3 = \lambda\, p(3) = 0.3$. First, we compute columns (2), (3), and (4). The nonzero entries are Poisson probabilities. Then we obtain the convolution of the p.f.'s in columns (2) and (3) and record the result in column (5). Finally, we convolute the p.f.'s displayed in columns (4) and (5) and record the result in column (6).

Remember that the complete p.f. is not displayed in either set of calculations. The example required probabilities for only $x = 0, 1, \ldots, 6$. However, $\Pr(S \leq 6) = f_S(0) + f_S(1) + \cdots + f_S(6) = 0.973526$. ▼

Formula (12.4.7) and Theorem 12.4.2 have another implication. Instead of defining a compound Poisson distribution of S by specifying the parameter λ and the d.f. $P(x)$ of the discrete individual claim amounts, we can define the distribution in terms of the possible individual claim amounts $x_1, x_2, \ldots, x_m$ and the parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ of the associated Poisson distributions described in part (b) of Theorem 12.4.2. Thus for x_i there is an associated Poisson distribution of N_i with parameter λ_i. In terms of this new definition of the distribution of S, we have from $\mathrm{E}[N_i] = \mathrm{Var}(N_i) = \lambda_i$ and the independence of the N_i's that

$$\mathrm{E}[S] = \mathrm{E}\left[\sum_{i=1}^{m} x_i N_i\right] = \sum_{i=1}^{m} x_i \lambda_i \tag{12.4.13}$$

and

$$\mathrm{Var}(S) = \mathrm{Var}\left(\sum_{i=1}^{m} x_i N_i\right) = \sum_{i=1}^{m} x_i^2 \lambda_i. \tag{12.4.14}$$

Formula (12.4.13) could be obtained by starting from (12.3.2) and noting that

$$\lambda p_1 = \lambda \sum_{i=1}^{m} x_i \pi_i = \sum_{i=1}^{m} x_i \lambda_i .$$

Similarly we can obtain (12.4.14) from (12.3.3).

In some cases it is useful, as in Example 12.4.1, to regard S as a sum of mutually independent random variables $x_i N_i$, $i = 1, 2, \ldots, m$, where $x_i N_i$ has a compound Poisson distribution with parameter λ_i and degenerate claim amount distribution at x_i. This interpretation follows from Theorem 12.4.2 and underlies the alternative method illustrated in Example 12.4.2.

There is a third way, the ***recursive method,*** for evaluating certain compound distributions for which the only possible claim amounts are positive integers. It is based on the recursive formula of the following theorem.

Theorem 12.4.3

For compound distributions where the probability distribution for N, the number of claims, satisfies the condition that $\Pr(N = n) = \Pr(N = n-1)\left(a + \frac{b}{n}\right)$ for $n = 1, 2, \ldots$ and where the distribution of claim amounts is restricted to the positive integers,

$$g(0) = \sum_{x=1}^{\infty} \left(a + \frac{b}{x}\right) p(x) g(0), \quad n = 1, 2, \ldots \qquad (12.4.15)$$

with the starting value given by $g(0) = \Pr(N = 0)$.

We establish the following lemma to be used in the proof of the theorem.

Lemma

For $X_1, X_2, \ldots, X_n$, which are independent and identically distributed random variables taking on values restricted to the positive integers, we have, for positive integer values of x,

(i) $$p^{*n}(x) = \sum_{i=1}^{x} p(i)\, p^{*(n-1)}(x-i),$$

(ii) $$p^{*n}(x) = \frac{n-1}{x} \sum_{i=1}^{x} i\, p(i)\, p^{*(n-1)}(x-i).$$

Proof of the Lemma:

For $n = 1$, both (i) and (ii) reduce to $p^{*1}(x) = p(x) \times p^{*(0)}(0)$. For $n > 1$ we establish (i) by using the Law of Total Probability to evaluate $\Pr(X_1 + X_2 + \cdots + X_n = x)$ by conditioning on the value taken by X_1 as

$$\sum_{i=1}^{x} \Pr(X_1 = i) \Pr(X_2 + X_3 + \cdots + X_n = x - i).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x-i)$ and $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n-1)$-fold and n-fold convolutions, respectively, of $p(i)$; see (2.3.4).

For $n > 1$, we establish (ii) by considering the conditional expectations $\mathrm{E}[X_k|X_1+X_2+X_3+\cdots+X_n = x]$ for $k = 1, 2, 3, \ldots, n$. From reasons of symmetry, these quantities are the same for all such k. Since their sum is x, each is equal to x/n. The conditional expectation $\mathrm{E}[X_1|X_1+X_2+X_3+\cdots+X_n = x]$ is evaluated as

$$\sum_{i=1}^{x} i \Pr(X_1 = i) \Pr(X_2+X_3+\cdots+X_n = x-i) / \Pr(X_1 + X_2 + X_3+\cdots+X_n = x).$$

We then note that $\Pr(X_2+X_3+\cdots+X_n = x-i)$ and that $\Pr(X_1+X_2+\cdots+X_n = x)$ can be evaluated by using $(n-1)$-fold and n-fold convolutions, respectively, of $p(i)$. Solving for $p^{*n}(x)$ completes the proof. ■

Proof of the Theorem:

First,

$$f_S(x) = \sum_{n=1}^{\infty} \Pr(N = n)\, p^{*n}(x).$$

With $\Pr(N = n) = [a + (b/n)] \Pr(N = n-1)$, we have

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n-1)\, p^{*n}(x) + \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n-1)\, p^{*n}(x),$$

and by the two parts of the lemma

$$f_S(x) = a \sum_{n=1}^{\infty} \Pr(N = n-1) \sum_{i=1}^{x} p(i)\, p^{*(n-1)}(x-i)$$

$$+ \sum_{n=1}^{\infty} \frac{b}{n} \Pr(N = n-1) \frac{n}{x} \sum_{i=1}^{x} i\, p(i)\, p^{*(n-1)}(x-i).$$

Interchanging the order of summation, we get

$$f_S(x) = a \sum_{i=1}^{x} p(i) \sum_{n=1}^{\infty} \Pr(N = n-1)\, p^{*(n-1)}(x-i)$$

$$+ \frac{b}{x} \sum_{i=1}^{x} ip(i) \sum_{n=1}^{\infty} \Pr(N = n-1)\, p^{*(n-1)}(x-i)$$

$$= a \sum_{i=1}^{x} p(i)\, f_S(x-i) + \frac{b}{x} \sum_{i=1}^{x} i\, p(i)\, f_S(x-i)$$

$$= \sum_{i=1}^{x} \left(a + \frac{b\,i}{x}\right) p(i)\, f_S(x-i).$$

■

We now examine the only three distributions that satisfy the required relationship between successive values of $\Pr(N = n)$.

a. Poisson: $[\Pr(N = n)/\Pr(N = n-1)] = \lambda/n$. The recursion formula for the compound Poisson is

$$f_S(x) = \frac{\lambda}{x} \sum_{i=1}^{x} ip(i) f_S(x - i) \quad \text{with} \quad f_S(0) = e^{-\lambda}. \tag{12.4.16}$$

b. Negative binomial: $[\Pr(N = n) / \Pr(N = n - 1)] = (1 - p)[(n + r - 1) / n]$ so that $a = (1 - p)$ and $b = (1 - p)(r - 1)$. The recursion formula for the compound negative binomial is

$$f_S(x) = (1 - p) \sum_{i=1}^{x} \left[(r - 1) \frac{i}{x} + 1 \right] p(i) f_S(x - i) \tag{12.4.17}$$

with $f_S(0) = p^r$.

c. Binomial with parameters m and p:

$$\frac{\Pr(N = n)}{\Pr(N = n - 1)} = \frac{m + 1 - n}{n} \frac{p}{1 - p}$$

so that $a = -[p / (1 - p)]$ and $b = (m + 1)[p / (1 - p)]$. The recursion formula in this case is

$$f_S(x) = \left(\frac{p}{1 - p} \right) \sum_{i=1}^{x} \left[(m + 1) \frac{i}{x} - 1 \right] p(i) f_S(x - i) \tag{12.4.18}$$

with $f_S(0) = (1 - p)^m$.

Example 12.4.2 (recomputed)

For the compound Poisson distribution of this example, compute $f_S(x) = \Pr(S = x)$ by the recursive method.

Solution:

Substituting the values used for the alternative method of calculation into (12.4.16) yields

$$f_S(x) = \frac{1}{x} [0.2 f_S(x - 1) + 0.6 f_S(x - 2) + 0.9 f_S(x - 3)] \qquad x = 1, 2, \ldots.$$

Recalling that $f_S(x) = 0$, $x < 0$, and $f_S(0) = e^{-\lambda} = 0.449329$, we readily reproduce the values of $f_S(x)$ given in the basic method calculations. ▼

12.5 Approximations to the Distribution of Aggregate Claims

In Section 2.4 the normal distribution was employed as an approximation to the distribution of aggregate claims in the individual model. The normal approximation is the first developed here for use with the collective model.

For the compound Poisson distribution, the two parameters of the normal approximation are given by (12.3.2) and (12.3.3). For the compound negative binomial distribution the parameters are given by (12.3.10) and (12.3.11). In each of the two cases the approximation is better when the expected number of claims is large, or, in other words, when λ is large for the compound Poisson case and when r is large

for the negative binomial. These two results are contained in Theorem 12.5.1, which may be interpreted as a version of the central limit theorem.

Theorem 12.5.1

(a) If S has a compound Poisson distribution specified by λ and $P(x)$, then the distribution of

$$Z = \frac{S - \lambda p_1}{\sqrt{\lambda p_2}} \tag{12.5.1}$$

converges to the standard normal distribution as $\lambda \to \infty$.

(b) If S has a compound negative binomial distribution specified by r, p, and $P(x)$, then the distribution of

$$Z = \frac{S - rp_1/p}{\sqrt{r(p_2/p^2 + p_1^2/p^2)}} \tag{12.5.2}$$

converges to the standard normal distribution as $r \to \infty$.

Proof:

We shall prove statement (a) by showing that

$$\lim_{\lambda \to \infty} M_Z(t) = e^{t^2/2}.$$

Statement (b) can be proved using a similar strategy, but the proof involves additional steps.

From (12.5.1), it follows that

$$M_Z(t) = M_S\left(\frac{t}{\sqrt{\lambda p_2}}\right) \exp\left(-\frac{\lambda p_1 t}{\sqrt{\lambda p_2}}\right).$$

Now we use (12.3.5) to obtain

$$M_Z(t) = \exp\left\{\lambda\left[M_X\left(\frac{t}{\sqrt{\lambda p_2}}\right) - 1\right] - \frac{\lambda p_1 t}{\sqrt{\lambda p_2}}\right\}, \tag{12.5.3}$$

and then substitute the expansion

$$M_X(t) = 1 + \frac{p_1 t}{1!} + \frac{p_2 t^2}{2!} + \cdots, \tag{12.5.4}$$

with $t/\sqrt{\lambda p_2}$ in place of t, into (12.5.3) to obtain

$$M_Z(t) = \exp\left(\frac{1}{2}t^2 + \frac{1}{6}\frac{1}{\sqrt{\lambda}}\frac{p_3}{p_2^{3/2}}t^3 + \cdots\right). \tag{12.5.5}$$

Then as $\lambda \to \infty$, $M_Z(t)$ approaches $e^{t^2/2}$, the m.g.f. of the standard normal distribution. ■

A normal distribution may not be the best approximation to the aggregate claim distribution because the normal distribution is symmetric and the distribution of

aggregate claims is often skewed. This skewness is evident in Table 12.5.1, which shows that the third central moment of S under each of the compound Poisson and compound negative binomial distributions is not 0. For positive claim amount distributions, $P(0) = 0$, and the third central moment of S is positive in each case.

Calculation of Third Central Moment of S

	Distribution of S	
Step	**Compound Poisson**	**Compound Negative Binomial**
$M_S(t)$	$\exp\{\lambda[M_X(t) - 1]\}$	$\left[\frac{p}{1 - qM_X(t)}\right]^r$
$\log M_S(t)$	$\lambda[M_X(t) - 1]$	$r \log p - r \log[1 - qM_X(t)]$
$\frac{d^3}{dt^3} \log M_S(t)$	$\lambda M'''_X(t)$	$\frac{rqM'''_X(t)}{1 - qM_X(t)} + \frac{3rq^2M'_X(t)M''_X(t)}{[1 - qM_X(t)]^2} + \frac{2rq^3M'_X(t)^3}{[1 - qM_X(t)]^3}$
$\mathrm{E}[(S - \mathrm{E}[S])^3] = \frac{d^3}{dt^3} \log M_S(t)\Big\vert_{t=0}$ *	λp_3	$\frac{rqp_3}{p} + \frac{3rq^2p_1p_2}{p^2} + \frac{2rq^3p_1^3}{p^3}$

*For $k \geq 4$, $\frac{d^k}{dt^k} \log M(t)\Big\vert_{t=0}$ is not the k-th central moment.

In completing Table 12.5.1 we use properties of the logarithm of a m.g.f., for example,

$$\frac{d}{dt} \log M_X(t)\Big|_{t=0} = \frac{M'_X(0)}{M_X(0)} = \mu$$

and

$$\frac{d^2}{dt^2} \log M_X(t)|_{t=0} = \frac{M''_X(0)M_X(0) - M'_X(0)^2}{M_X(0)^2} = \sigma^2.$$

In Exercise 12.23(a), the reader is asked to confirm the relation used in the last row of Table 12.5.1.

Because of this skewness we seek a more general approximation to the distribution of aggregate claims, one that accommodates skewness. For this second approximation, we begin with a gamma distribution. This choice is motivated by the fact that the gamma distribution has a positive third central moment as do the compound Poisson and compound negative binomial distributions with positive claim amounts. We let $G(x{:}\alpha, \beta)$ denote the d.f. of the gamma distribution with parameters α and β; that is,

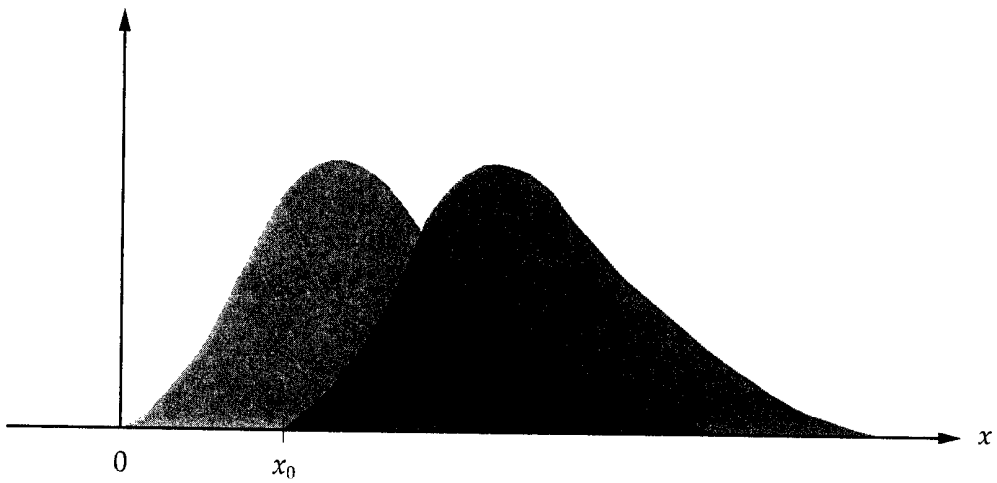
$$G(x{:}\alpha, \beta) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}\, dt. \tag{12.5.6}$$

Then for any x_0 we define a new d.f., denoted by $H(x{:}\alpha, \beta, x_0)$, as

$$H(x{:}\alpha, \beta, x_0) = G(x - x_0{:}\alpha, \beta). \tag{12.5.7}$$

This amounts to a translation of the distribution $G(x{:}\alpha, \beta)$ by x_0. Figure 12.5.1 illustrates this for the case $x_0 > 0$ where $g(x)$, $x \geq 0$, and $h(x)$, $x \geq x_0$, denote, respectively, the p.d.f.'s associated with $G(x{:}\alpha, \beta)$ and $H(x{:}\alpha, \beta, x_0)$.

Translated Gamma Distribution



We approximate the distribution of aggregate claims S by a translated gamma distribution where the parameters α, β, and x_0 are selected by equating the first moment and second and third central moments of S with the corresponding characteristics of the translated gamma distribution. Since central moments of the translated gamma are the same as for the basic gamma distribution, this procedure imposes the requirements

$$\mathrm{E}[S] = x_0 + \frac{\alpha}{\beta}, \tag{12.5.8}$$

$$\mathrm{Var}(S) = \frac{\alpha}{\beta^2}, \tag{12.5.9}$$

$$\mathrm{E}[(S - \mathrm{E}[S])^3] = \frac{2\alpha}{\beta^3}. \tag{12.5.10}$$

From these we obtain

$$\beta = 2\,\frac{\mathrm{Var}(S)}{\mathrm{E}[(S - \mathrm{E}[S])^3]}, \tag{12.5.11}$$

$$\alpha = 4\,\frac{[\mathrm{Var}(S)]^3}{\mathrm{E}[(S - \mathrm{E}[S])^3]^2}, \tag{12.5.12}$$

$$x_0 = \mathrm{E}[S] - 2\,\frac{[\mathrm{Var}(S)]^2}{\mathrm{E}[(S - \mathrm{E}[S])^3]}. \tag{12.5.13}$$

For a compound Poisson distribution this procedure leads to

$$\alpha = 4\lambda \frac{p_2^3}{p_3^2}, \tag{12.5.14}$$

$$\beta = 2 \frac{p_2}{p_3}, \tag{12.5.15}$$

$$x_0 = \lambda p_1 - 2\lambda \frac{p_2^2}{p_3}. \tag{12.5.16}$$

Remark:

We can show that if $\alpha \to \infty$, $\beta \to \infty$, and $x_0 \to -\infty$ such that

$$x_0 + \frac{\alpha}{\beta} = \mu \text{ (constant)}, \qquad \frac{\alpha}{\beta^2} = \sigma^2 \text{ (constant)}, \tag{12.5.17}$$

the distribution $H(x{:}\alpha, \beta, x_0)$ converges to the $N(\mu, \sigma^2)$ distribution. Therefore, the family of normal distributions is contained, as limiting distributions, within this family of three-parameter gamma distributions. In this sense, this approximation is a generalization of the normal approximation.

Example 12.5.1

Consider the Poisson distribution with parameter $\lambda = 16$. This is the same as the compound Poisson distribution with $\lambda = 16$ and a degenerate claim amount distribution at 1. Compare this distribution with approximations by

a. A translated gamma distribution
b. A normal distribution.

Solution:

a. Here $p_k = 1$, $k = 1, 2, 3$, and from (12.5.14)–(12.5.16), we have $\alpha = 64$, $\beta = 2$, $x_0 = -16$. Note that, unlike the case in Figure 12.5.1, x_0 is negative.
b. For the normal approximation, we use $\mu = 16$ and $\sigma = 4$.

The results given below compare the three distributions. In the approximations, the half-integer discontinuity correction was used to approximate $F_S(x)$ for $x = 5, 10, \ldots, 40$.

	Exact	Approximations	
x	$\sum_{y=0}^{x} \frac{e^{-16}(16)^y}{y!}$	$G(x+16.5:64, 2)$	$\Phi\left(\frac{x+0.5-16}{4}\right)$
5	0.001384	0.001636	0.004332
10	0.077396	0.077739	0.084566
15	0.466745	0.466560	0.450262
20	0.868168	0.868093	0.869705
25	0.986881	0.986604	0.991226
30	0.999433	0.999378	0.999856
35	0.999988	0.999985	0.999999
40	1.000000	1.000000	1.000000

▼

In the case of the compound negative binomial distribution there is an additional argument that supports the use of a gamma approximation. The argument is outlined in the Appendix to this chapter.

12.6 Notes and References

Chapter 2 of Seal (1969) contains an extensive survey of the literature on collective risk models, including the pioneering work of Lundberg on the compound Poisson distribution. Several authors, for example, Dropkin (1959) and Simon (1960), have used the negative binomial distribution to model the number of automobile accidents by a collection of policyholders in a fixed period.

In Example 12.3.1 we derived the negative binomial distribution by assuming that the unknown Poisson parameter has a gamma distribution. This idea goes back at least as far as Greenwood and Yule's work on accident proneness (1920). An alternative derivation in terms of a contagion model is due to Polya and Eggenberger and may be found in Chapter 2 of Bühlmann (1970). In the special case where r is an integer, the negative binomial can be obtained as the distribution of the number of Bernoulli trials that end in failure prior to the r-th success. This development may be found in most probability texts, but has little relevance to the subject of this chapter.

Theorem 12.4.2 has been known for some time and can be studied in Section 2, Chapter 2 of Feller (1968). The alternative method for computing probabilities for a compound Poisson distribution, which is based on Theorem 12.4.2, was suggested by Pesonen (1967) and implemented by Halmstad (1976) in the calculation of stop-loss premiums. Theorem 12.4.2 has a converse, which was not stated in Section 12.4. Renyi (1962) shows that the mutual independence of $N_1, N_2, \ldots, N_m$ implies that N has a Poisson distribution. Hence the alternative method of computing will work only for the compound Poisson distribution.

The alternative method of computation may also be adopted to build a simulation model for aggregate claims. Instead of determining the individual claim amounts, one simulates $N_1, N_2, \ldots, N_m$ and obtains a realization of S directly from (12.4.7). For one determination of S, the expected number of random numbers required is $1 + \lambda$ under the basic method. If the alternative method is used, exactly m random numbers are required for each determination of a value of S.

There are several more elaborate methods of approximating the distribution of aggregate claims. The normal power and Esscher approximations are described in Beard, Pesonen, and Pentikäinen (1984). Several of the approximation methods have been compared by Bohman and Esscher (1963, 1964). Seal (1978a) presents the case for the translated gamma approximation and illustrates its excellent performance. Bowers (1966) approximated the distribution of aggregate claims by a sum of orthogonal functions, the first term of which is the gamma distribution. The result, stated in the Appendix to this chapter, that the gamma distribution can be

obtained as a limit from the compound negative binomial is due to Lundberg (1940).

Sometimes the distribution of aggregate claims can be obtained from a numerical inversion of its m.g.f.; this is developed in Chapter 3 of Seal (1978).

A monograph by Panjer and Willmot (1992) develops more completely the ideas of this chapter with particular emphasis on recursive calculation and discrete approximations.

Appendix

Theorem 12.A.1

If the random variables S_k, $k = 0, 1, 2, \ldots$, have compound negative binomial distributions with parameters r and $q(k)$ and claim amount distribution $P(x)$, and if the parameters of the negative binomial distributions are such that

$$\frac{q(k)}{p(k)} = k\frac{q}{p}$$

for $k = 1, 2, 3, \ldots$, where $q = 1 - p$ is a constant, then the distribution of

$$\frac{S_k}{\mathrm{E}[S_k]}$$

approaches $G(x; r, r)$ as $k \to \infty$.

Proof:

Using (12.3.12), we find the m.g.f. of $S_k / \mathrm{E}[S_k]$ to be

$$\left[\frac{p(k)}{1 - q(k)M_X(t / \mathrm{E}[S_k])}\right]^r. \tag{12.A.1}$$

We also have

$$M_X\left(\frac{t}{\mathrm{E}[S_k]}\right) = 1 + \frac{p_1}{\mathrm{E}[S_k]}\,t + \frac{p_2}{2\mathrm{E}[S_k]^2}\,t^2 + \cdots. \tag{12.A.2}$$

If (12.A.2) is substituted into (12.A.1), we obtain

$$\left\{\frac{p(k)}{1 - q(k) - [q(k)p_1 / \mathrm{E}[S_k]]t - [q(k)p_2 / 2\mathrm{E}[S_k]^2]t^2 - \cdots}\right\}^r. \tag{12.A.3}$$

Now, since

$$\mathrm{E}[S_k] = r\,\frac{q(k)p_1}{p(k)} = r\,\frac{kqp_1}{p},$$

we see that the m.g.f. of $S_k / \mathrm{E}[S_k]$ is

$$\left[1 - \frac{1}{r}t - \frac{p_2}{2r^2p_1^2(q/p)k}t^2 - \cdots\right]^{-r} = \left[1 - \frac{1}{r}t - R(k)\right]^{-r}$$

where the remainder term $R(k)$ is such that $\lim_{k\to\infty} R(k) = 0$. Therefore,

$$\lim_{k\to\infty} \mathrm{E}\left[\exp\left(t\frac{S_k}{\mathrm{E}[S_k]}\right)\right] = \left(\frac{r}{r-t}\right)^r, \tag{12.A.4}$$

which is the m.g.f. of a $G(x{:}r, r)$ distribution. ■

It follows from (12.A.4) that the m.g.f. of S_k itself is approximately

$$\left(\frac{r}{r - \mathrm{E}[S_k]t}\right)^r = \left\{\frac{r}{r - [rq(k)/p(k)]p_1t}\right\}^r = \left\{\frac{p(k)/[q(k)p_1]}{p(k)/[q(k)p_1] - t}\right\}^r,$$

which is the m.g.f. of $G\{x{:}r, [p(k)/q(k)p_1]\}$. Thus, when k is large, which under the hypothesis of Theorem 12.A.1 implies that the expected number of claims, $rq(k)/p(k) = rk(q/p)$, is large, the distribution of aggregate claims is approximately a gamma distribution.

Theorem 12.A.1 is presented to provide an argument supporting the use of gamma distributions to approximate the distribution of aggregate claims. Comparison of the main ideas in Theorem 12.5.1(b) and Theorem 12.A.1 leads to insights. Theorem 12.5.1(b) follows closely the pattern of the central limit theorem. If in (12.5.2) one writes

$$Z = \frac{S/r - (q/p)p_1}{\sqrt{(q/p)p_2 + (q^2/p^2)p_1^2}/\sqrt{r}},$$

the correspondence is clear, with the parameter r playing the role of n in the central limit theorem.

In Theorem 12.A.1 the parameter r of the negative binomial distribution remains fixed. The expected number of claims changes in proportion to a size parameter k, by compensating changes in $q(k)$ and $p(k) = 1 - q(k)$. Under the hypothesis of Theorem 12.A.1,

$$\mathrm{Var}(S_k) = \frac{rkq}{p}p_2 + r\frac{k^2q^2}{p^2}p_1^2$$

and

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) = \frac{p}{rkqp_1^2}p_2 + \frac{1}{r}.$$

As the size parameter $k \to \infty$,

$$\mathrm{Var}\left(\frac{S_k}{\mathrm{E}[S_k]}\right) \to \frac{1}{r},$$

as indicated by Theorem 12.A.1. Thus the gamma approximation may be considered in the negative binomial case when the expected number of claims is large and the claim amount distribution has relatively small dispersion.

Exercises

Section 12.1

12.1. Let S denote the number of people crossing a certain intersection by car in a given hour. How would you model S as a random sum?

12.2. Let S denote the total amount of rain that falls at a weather station in a given month. How would you model S as a random sum?

Section 12.2

12.3. Suppose N has a binomial distribution with parameters n and p. Express each of the following in terms of n, p, p_1, p_2, and $M_X(t)$:
a. E[S] b. Var(S) c. $M_S(t)$.

12.4. For the distribution specified in Example 12.2.2, calculate
a. E[N] b. Var(N) c. E[X]
d. Var(X) e. E[S] f. Var(S).

Section 12.3

12.5. Suppose that the claim amount distribution is the same as in Example 12.2.2, but that N has a Poisson distribution with E[N] = 1.7. Calculate
a. E[S] b. Var(S).

12.6. Suppose that S has a compound Poisson distribution with $\lambda = 2$ and $p(x) = 0.1x$, $x = 1, 2, 3, 4$. Calculate probabilities that aggregate claims equal 0, 1, 2, 3, and 4.

12.7. Consider the family of negative binomial distributions with parameters r and p. Let $r \to \infty$ and $p \to 1$ such that $r(1 - p) = \lambda$ remains constant. Show that the limit obtained is the Poisson distribution with parameter λ. [Hint: Note that $p^r = [1 - (\lambda / r)]^r \to e^{-\lambda}$ as $r \to \infty$, and consider the convergence of the m.g.f.]

12.8. Suppose that S has a compound Poisson distribution with Poisson parameter λ and claim amount p.f.

$$p(x) = [-\log(1 - c)]^{-1} \frac{c^x}{x} \qquad x = 1, 2, 3, \ldots, \qquad 0 < c < 1.$$

Consider the m.g.f. of S and show that S has a negative binomial distribution with parameters p and r. Express p and r in terms of c and λ.

12.9. Let

$$g(x) = 3^{18} x^{17} \frac{e^{-3x}}{17!}$$

and

$$h(x) = 3^6 x^5 \frac{e^{-3x}}{5!} \qquad x > 0$$

be two p.d.f.'s. Write the convolution of these two distributions, that is, exhibit $g*h(x)$. [Hint: Proceed directly from the definition of convolution in Section 2.3, or make use of (12.3.19).]

12.10. Suppose that the number of accidents incurred by an insured driver in a single year has a Poisson distribution with parameter λ. If an accident happens, the probability is p that the damage amount will exceed a deductible amount. On the assumption that the number of accidents is independent of the severity of the accidents, derive the distribution of the number of accidents that result in a claim payment.

12.11. The m.g.f. of the Poisson inverse Gaussian distribution is given in the solution to Example 12.3.2. Replace the α parameter by $\lambda\beta$ so that the mean is now λ and the variance is $\lambda + \lambda\beta$. Show that

$$\lim_{\beta \to \infty} M_N(t) = e^{\lambda(e^t - 1)}.$$

This confirms that the Poisson inverse Gaussian distribution approaches the Poisson distribution as $\beta \to \infty$ and the mean remains constant.

Section 12.4

12.12. Suppose that S_1 has a compound Poisson distribution with Poisson parameter $\lambda = 2$ and claim amounts that are 1, 2, or 3 with probabilities 0.2, 0.6, and 0.2, respectively. In addition, S_2 has a compound Poisson distribution with Poisson parameter $\lambda = 6$ and claim amounts that are either 3 or 4 with probability 0.5 for each. If S_1 and S_2 are independent, what is the distribution of $S_1 + S_2$?

12.13. Suppose that N_1, N_2, N_3 are mutually independent and that N_i has a Poisson distribution with $\mathrm{E}[N_i] = i^2$, $i = 1, 2, 3$. What is the distribution of $S = -2N_1 + N_2 + 3N_3$?

12.14. If N has a Poisson distribution with parameter λ, express $\Pr(N = n + 1)$ in terms of $\Pr(N = n)$.

Note that this recursive formula may be useful in calculations such as those for successive entries in columns (2), (3), and (4) of the alternate method calculations of Example 12.6.

12.15. Suppose that S has a compound Poisson distribution with parameter λ and discrete p.f. $p(x)$, $x > 0$. Let $0 < \alpha < 1$.

Consider $\tilde{S}$ with a distribution that is compound Poisson with Poisson parameter $\tilde{\lambda} = \lambda / \alpha$ and claim amount p.f. $\tilde{p}(x)$ where

$$\tilde{p}(x) = \begin{cases} \alpha p(x) & x > 0 \\ 1 - \alpha & x = 0. \end{cases}$$

This means we are allowing for claim amounts of 0 (as could happen if there is a deductible) and are modifying the distributions accordingly. Show that S and $\tilde{S}$ have the same distribution by

a. Comparing the m.g.f.'s of S and $\tilde{S}$
b. Comparing the definition of the distribution of S and $\tilde{S}$ in terms of possible claim amounts and the Poisson parameters of the distributions of their frequencies.

12.16. In Example 12.2.2, let N_1 be the random number of claims of amount 1 and N_2 the random number of claims of amount 2. Compute

a. $\Pr(N_1 = 1)$ b. $\Pr(N_2 = 1)$ c. $\Pr(N_1 = 1, N_2 = 1)$.

Are N_1 and N_2 independent?

12.17. Compute $f_S(x)$ for $x = 0, 1, 2, \ldots, 5$ for the following three compound distributions, each with claim amount distribution given by $p(1) = 0.7$ and $p(2) = 0.3$:

a. Poisson with $\lambda = 4.5$
b. Negative binomial with $r = 4.5$ and $p = 0.5$
c. Binomial with $m = 9$ and $p = 0.5$
d. For each of the distributions of (a), (b), and (c) calculate the mean and variance of the number of claims.

12.18. Let S, as given in (12.4.7), have a compound negative binomial distribution with parameters r and p and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).

a. Show that N_i has a negative binomial distribution with parameters r and $p/(p + q\pi_i)$.
b. Show that, in general, N_1 and N_2 are not independent.
[Hint: Use the joint m.g.f. of $N_1, N_2, \ldots, N_m$ as in the proof of Theorem 12.4.2.]

12.19. Show that the compound distribution of Example 12.2.2 does not satisfy the hypotheses of Theorem 12.4.3.

Section 12.5

12.20. Show that if N has a Poisson distribution with parameter λ, the distribution of

$$Z = \frac{N - \lambda}{\sqrt{\lambda}}$$

approaches a $N(0, 1)$ distribution as $\lambda \to \infty$.

12.21. Use $\log M_S(t)$ as given in Table 12.5.1 to verify (12.3.3) and (12.3.11).

12.22. Suppose that the d.f. of S is $G(x{:}\alpha, \beta)$. Use the m.g.f. [see (12.3.19)] to show that

$$\mathrm{E}[S^h] = \frac{\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + h - 1)}{\beta^h} \qquad h = 1, 2, 3, \ldots.$$

12.23. a. Verify that

$$\frac{d^3}{dt^3} \log M_X(t)\bigg|_{t=0} = \mathrm{E}[(X - \mathrm{E}[X])^3].$$

b. Use (a) to show that if S has a $G(x{:}\alpha, \beta)$ distribution, then

$$\mathrm{E}[(S - \mathrm{E}[S])^3] = \frac{2\alpha}{\beta^3}.$$

12.24. a. For a given α, determine β and x_0 so that $H(x{:}\alpha, \beta, x_0)$ has mean 0 and variance 1.
b. What is the limit of $H(x{:}\alpha, \sqrt{\alpha}, -\sqrt{\alpha})$ as $\alpha \to \infty$?

12.25. Suppose that S has a compound Poisson distribution with $\lambda = 12$ and claim amounts that are uniformly distributed between 0 and 1. Approximate $\Pr(S < 10)$ using
a. The normal approximation
b. The translated gamma approximation.

Miscellaneous

12.26. The loss ratio for a collection of insurance policies over a single premium period is defined as $R = S / G$ where S is aggregate claims and G is aggregate premiums. Assume that $G = p_1\mathrm{E}[N](1 + \theta)$, $\theta > 0$.
a. Show that

$$\mathrm{E}[R] = (1 + \theta)^{-1}$$

and that

$$\mathrm{Var}(R) = \frac{\mathrm{E}[N]\ \mathrm{Var}(X) + p_1^2\ \mathrm{Var}(N)}{[p_1\ \mathrm{E}[N](1 + \theta)]^2}.$$

b. Develop an expression for $\mathrm{Var}(R)$ if
(i) N has a Poisson distribution
(ii) N has a negative binomial distribution.

12.27. Suppose that the distribution of S_1 is compound Poisson, given by λ and $P_1(x)$, and that the distribution of S_2 is compound negative binomial, given by r, p with $q = 1 - p$, and $P_2(x)$. Show that S_1 and S_2 have the same distribution provided that $\lambda = -r \log p$ and

$$P_1(x) = \frac{\sum_{k=1}^{\infty} (q^k / k)\ P_2^{*k}(x)}{-\log p}.$$

[Hint: Show equality of the m.g.f.'s.]
Note that, in the sense of this exercise, every compound negative binomial distribution can be considered as compound Poisson.

12.28. Let S, as given in (12.4.7), have a compound Poisson inverse Gaussian distribution with parameters α and β and p.f. of claim amounts given by the discrete p.f. exhibited in (12.4.6).
a. Show that N_i has a Poisson inverse Gaussian distribution and determine its parameter values.
b. Show that, in general, N_1 and N_2 are not independent.

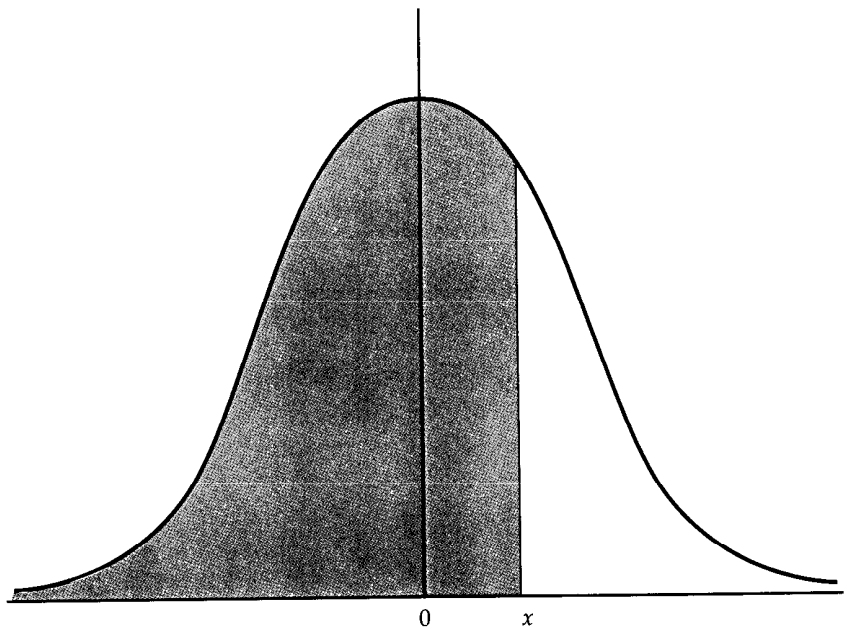
12.29. Follow the steps displayed in Table 12.5.1 to show that the third central moment of S, when it has a compound Poisson inverse Gaussian distribution, can be expressed in the parameters of its distribution as

$$\frac{\alpha}{\beta} p_3 + \frac{3\alpha}{\beta^2} p_1 p_2 + \frac{3\alpha}{\beta^3} p_1^3.$$

12.30. a. Verify that the extension of (2.2.10) for the mean and (2.2.11) for the variance to the third central moment, $\mu_3(W) = E[\{W - E[W]\}^3]$, is $E[\mu_3(W|V)] + 3\,\mathrm{Cov}(\mathrm{Var}(W|V), E[W|V]) + \mu_3(E[W|V])$.
[Hint: Write $E[W - E[W]] = E[(W - E[W|V]) + (E[W|V] - E[W])]$, expand the third power, and take expectations termwise.]
b. Apply the result of (a) to S of (12.1.1) to express its third central moment in terms of the parameters of its distribution. [Hint: Refer to (12.2.5) and (12.2.6).]
c. Apply the formula of (b) to the compound distributions of Table 12.5.1 to confirm the third central moments shown there.
d. Apply the formula of (b) to the compound Poisson inverse Gaussian distribution to confirm the formula of the previous exercise.

Appendix 1

NORMAL DISTRIBUTION TABLE



Normal Distribution

The table on page 674 gives the value of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-w^2/2}\, dw$$

for certain values of x. The integer of x is given in the top row, and the first decimal place of x is given in the left column. Since the density function of x is symmetric, the value of the cumulative distribution function for negative x can be obtained by subtracting from unity the value of the cumulative distribution function for x.

x	0	1	2	3
0.0	0.5000	0.8413	0.9772	0.9987
0.1	0.5398	0.8643	0.9821	0.9990
0.2	0.5793	0.8849	0.9861	0.9993
0.3	0.6179	0.9032	0.9893	0.9995
0.4	0.6554	0.9192	0.9918	0.9997
0.5	0.6915	0.9332	0.9938	0.9998
0.6	0.7257	0.9452	0.9953	0.9998
0.7	0.7580	0.9554	0.9965	0.9999
0.8	0.7881	0.9641	0.9974	0.9999
0.9	0.8159	0.9713	0.9981	1.0000

Selected Points of the Normal Distribution

$\Phi(x)$	x
0.800	0.842
0.850	1.036
0.900	1.282
0.950	1.645
0.975	1.960
0.990	2.326
0.995	2.576

Appendix 2A

ILLUSTRATIVE LIFE TABLE

Illustrative Life Table: Basic Functions

Age	l_x	d_x	1,000 q_x
0	100 000.00	2 042.1700	20.4217
1	97 957.83	131.5672	1.3431
2	97 826.26	119.7100	1.2237
3	97 706.55	109.8124	1.1239
4	97 596.74	101.7056	1.0421
5	97 495.03	95.2526	0.9770
6	97 399.78	90.2799	0.9269
7	97 309.50	86.6444	0.8904
8	97 222.86	84.1950	0.8660
9	97 138.66	82.7816	0.8522
10	97 055.88	82.2549	0.8475
11	96 973.63	82.4664	0.8504
12	96 891.16	83.2842	0.8594
13	96 807.88	84.5180	0.8730
14	96 723.36	86.0611	0.8898
15	96 637.30	87.7559	0.9081
16	96 549.54	89.6167	0.9282
17	96 459.92	91.6592	0.9502
18	96 368.27	93.9005	0.9744
19	96 274.36	96.3596	1.0009
20	96 178.01	99.0569	1.0299
21	96 078.95	102.0149	1.0618
22	95 976.93	105.2582	1.0967
23	95 871.68	108.8135	1.1350
24	95 762.86	112.7102	1.1770
25	95 650.15	116.9802	1.2330
26	95 533.17	121.6585	1.2735
27	95 411.51	126.7830	1.3288
28	95 284.73	132.3953	1.3895
29	95 152.33	138.5406	1.4560

Illustrative Life Table: Basic Functions

Age	l_x	d_x	1,000 q_x
30	95 013.79	145.2682	1.5289
31	94 868.53	152.6317	1.6089
32	94 715.89	160.6896	1.6965
33	94 555.20	169.5052	1.7927
34	94 385.70	179.1475	1.8980
35	94 206.55	189.6914	2.0136
36	94 016.86	201.2179	2.1402
37	93 815.64	213.8149	2.2791
38	93 601.83	227.5775	2.4313
39	93 374.25	242.6085	2.5982
40	93 131.64	259.0186	2.7812
41	92 872.62	276.9271	2.9818
42	92 595.70	296.4623	3.2017
43	92 299.23	317.7619	3.4427
44	91 981.47	340.9730	3.7070
45	91 640.50	366.2529	3.9966
46	91 274.25	393.7687	4.3141
47	90 880.48	423.6978	4.6621
48	90 456.78	456.2274	5.0436
49	90 000.55	491.5543	5.4617
50	89 509.00	529.8844	5.9199
51	88 979.11	571.4316	6.4221
52	88 407.68	616.4165	6.9724
53	87 791.26	665.0646	7.5755
54	87 126.20	717.6041	8.2364
55	86 408.60	774.2626	8.9605
56	85 634.33	835.2636	9.7538
57	84 799.07	900.8215	10.6230
58	83 898.25	971.1358	11.5752
59	82 927.11	1 046.3843	12.6181
60	81 880.73	1 126.7146	13.7604
61	80 754.01	1 212.2343	15.0114
62	79 541.78	1 302.9994	16.3813
63	78 238.78	1 399.0010	17.8812
64	76 839.78	1 500.1504	19.5231
65	75 339.63	1 606.2618	21.3203
66	73 733.37	1 717.0334	23.2871
67	72 016.33	1 832.0273	25.4391
68	70 184.31	1 950.6476	27.7932
69	68 233.66	2 072.1177	30.3680

Illustrative Life Table: Basic Functions

Age	l_x	d_x	1,000 q_x
70	66 161.54	2 195.4578	33.1833
71	63 966.08	2 319.4639	36.2608
72	61 646.62	2 442.6884	39.6240
73	59 203.93	2 563.4258	43.2982
74	56 640.51	2 679.7050	47.3108
75	53 960.80	2 789.2905	51.6911
76	51 171.51	2 889.6965	56.4708
77	48 281.81	2 978.2164	61.6840
78	45 303.60	3 051.9717	67.3671
79	42 251.62	3 107.9833	73.5589
80	39 143.64	3 143.2679	80.3009
81	36 000.37	3 154.9603	87.6369
82	32 845.41	3 140.4624	95.6134
83	29 704.95	3 097.6146	104.2794
84	26 607.34	3 024.8830	113.6860
85	23 582.45	2 921.5530	123.8867
86	20 660.90	2 787.9129	134.9367
87	17 872.99	2 625.4088	146.8926
88	15 247.58	2 436.7474	159.8121
89	12 810.83	2 225.9244	173.7533
90	10 584.91	1 998.1533	188.7738
91	8 586.75	1 759.6818	204.9298
92	6 827.07	1 517.4869	222.2749
93	5 309.58	1 278.8606	240.8589
94	4 030.72	1 050.9136	260.7257
95	2 979.81	840.0452	281.9122
96	2 139.77	651.4422	304.4456
97	1 488.32	488.6776	328.3410
98	999.65	353.4741	353.5993
99	646.17	245.6772	380.2041
100	400.49	163.4494	408.1188
101	237.05	103.6560	437.2837
102	133.39	62.3746	467.6133
103	71.01	35.4358	498.9935
104	35.58	18.9023	531.2793
105	16.68	9.4105	564.2937
106	7.27	4.3438	597.8266
107	2.92	1.8458	631.6360
108	1.08	0.7163	665.4495
109	0.36	0.2517	698.9685
110	0.11	0.0793	731.8742

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
0	16.80096	49.0025	25.9210
1	17.09819	32.1781	8.8845
2	17.08703	32.8097	8.6512
3	17.07314	33.5957	8.5072
4	17.05670	34.5264	8.4443
5	17.03786	35.5930	8.4547
6	17.01675	36.7875	8.5310
7	16.99351	38.1031	8.6666
8	16.96823	39.5341	8.8553
9	16.94100	41.0757	9.0917
10	16.91187	42.7245	9.3712
11	16.88089	44.4782	9.6902
12	16.84807	46.3359	10.0460
13	16.81340	48.2981	10.4373
14	16.77685	50.3669	10.8638
15	16.73836	52.5459	11.3268
16	16.69782	54.8404	11.8295
17	16.65515	57.2558	12.3749
18	16.61024	59.7977	12.9665
19	16.56299	62.4720	13.6080
20	16.51330	65.2848	14.3034
21	16.46105	68.2423	15.0569
22	16.40614	71.3508	15.8730
23	16.34843	74.6170	16.7566
24	16.28783	78.0476	17.7128
25	16.22419	81.6496	18.7472
26	16.15740	85.4300	19.8657
27	16.08733	89.3962	21.0744
28	16.01385	93.5555	22.3802
29	15.93683	97.9154	23.7900
30	15.85612	102.4835	25.3113
31	15.77161	107.2676	26.9520
32	15.68313	112.2754	28.7206
33	15.59057	117.5148	30.6259
34	15.49378	122.9935	32.6772
35	15.39262	128.7194	34.8843
36	15.28696	134.7002	37.2574
37	15.17666	140.9437	39.8074
38	15.06159	147.4572	42.5455
39	14.94161	154.2484	45.4833

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
40	14.81661	161.3242	48.6332
41	14.68645	168.6916	52.0077
42	14.55102	176.3572	55.6199
43	14.41022	184.3271	59.4833
44	14.26394	192.6071	63.6117
45	14.11209	201.2024	68.0193
46	13.95459	210.1176	72.7205
47	13.79136	219.3569	77.7299
48	13.62235	228.9234	83.0624
49	13.44752	238.8198	88.7329
50	13.26683	249.0475	94.7561
51	13.08027	259.6073	101.1469
52	12.88785	270.4988	107.9196
53	12.68960	281.7206	115.0885
54	12.48556	293.2700	122.6672
55	12.27581	305.1431	130.6687
56	12.06042	317.3346	139.1053
57	11.83953	329.8381	147.9883
58	11.61327	342.6452	157.3280
59	11.38181	355.7466	167.1332
60	11.14535	369.1310	177.4113
61	10.90412	382.7858	188.1682
62	10.65836	396.6965	199.4077
63	10.40837	410.8471	211.1318
64	10.15444	425.2202	223.3401
65	9.89693	439.7965	236.0299
66	9.63619	454.5553	249.1958
67	9.37262	469.4742	262.8299
68	9.10664	484.5296	276.9212
69	8.83870	499.6963	291.4559
70	8.56925	514.9481	306.4172
71	8.29879	530.2574	321.7850
72	8.02781	545.5957	337.5361
73	7.75683	560.9339	353.6443
74	7.48639	576.2419	370.0803
75	7.21702	591.4895	386.8119
76	6.94925	606.6460	403.8038
77	6.68364	621.6808	421.0184
78	6.42071	636.5634	438.4155
79	6.16101	651.2639	455.9527

Illustrative Life Table: Single Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_x$	1,000 A_x	1,000 $({}^2A_x)$
80	5.90503	665.7528	473.5861
81	5.65330	680.0019	491.2698
82	5.40629	693.9837	508.9574
83	5.16446	707.6723	526.6012
84	4.92824	721.0431	544.1537
85	4.69803	734.0736	561.5675
86	4.47421	746.7428	578.7956
87	4.25710	759.0320	595.7923
88	4.04700	770.9244	612.5133
89	3.84417	782.4056	628.9163
90	3.64881	793.4636	644.9611
91	3.46110	804.0884	660.6105
92	3.28118	814.2726	675.8298
93	3.10914	824.0111	690.5878
94	2.94502	833.3007	704.8565
95	2.78885	842.1408	718.6115
96	2.64059	850.5325	731.8321
97	2.50020	858.4791	744.5010
98	2.36759	865.9853	756.6047
99	2.24265	873.0577	768.1330
100	2.12522	879.7043	779.0793
101	2.01517	885.9341	789.4400
102	1.91229	891.7573	799.2147
103	1.81639	897.1852	808.4054
104	1.72728	902.2295	817.0170
105	1.64472	906.9025	825.0563
106	1.56850	911.2170	832.5324
107	1.49838	915.1860	839.4558
108	1.43414	918.8224	845.8386
109	1.37553	922.1396	851.6944
110	1.32234	925.1507	857.0377

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $({}^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $({}^2A_{x:x+10})$
0	16.13448	86.7274	50.8875	16.28443	78.2400	34.7076
1	16.71842	53.6745	17.4565	16.55328	63.0218	18.1309
2	16.70637	54.3565	16.9753	16.52270	64.7527	18.2195
3	16.68957	55.3072	16.6683	16.48839	66.6947	18.4277
4	16.66839	56.5060	16.5191	16.45053	68.8378	18.7468
5	16.64317	57.9339	16.5121	16.40925	71.1745	19.1700
6	16.61421	59.5733	16.6324	16.36464	73.6996	19.6923
7	16.58178	61.4085	16.8664	16.31677	76.4091	20.3096
8	16.54614	63.4258	17.2017	16.26571	79.2997	21.0188
9	16.50749	65.6137	17.6271	16.21147	82.3696	21.8172
10	16.46599	67.9626	18.1330	16.15408	85.6181	22.7036
11	16.42178	70.4655	18.7116	16.09353	89.0457	23.6776
12	16.37492	73.1176	19.3572	16.02977	92.6543	24.7402
13	16.32547	75.9170	20.0661	15.96277	96.4469	25.8935
14	16.27340	78.8643	20.8373	15.89244	100.4282	27.1413
15	16.21865	81.9632	21.6726	15.81866	104.6042	28.4891
16	16.16111	85.2203	22.5769	15.74131	108.9826	29.9441
17	16.10065	88.6424	23.5556	15.66025	113.5710	31.5141
18	16.03715	92.2366	24.6142	15.57534	118.3771	33.2071
19	15.97049	96.0099	25.7588	15.48645	123.4087	35.0317
20	15.90053	99.9697	26.9958	15.39343	128.6737	36.9970
21	15.82715	104.1234	28.3320	15.29615	134.1800	39.1126
22	15.75021	108.4786	29.7746	15.19448	139.9353	41.3884
23	15.66958	113.0429	31.3311	15.08826	145.9474	43.8349
24	15.58511	117.8241	33.0098	14.97738	152.2240	46.4632
25	15.49667	122.8299	34.8192	14.86169	158.7725	49.2847
26	15.40413	128.0682	36.7681	14.74106	165.6003	52.3114
27	15.30734	133.5468	38.8662	14.61538	172.7144	55.5555
28	15.20617	139.2737	41.1234	14.48452	180.1217	59.0301
29	15.10047	145.2564	43.5502	14.34836	187.8286	62.7483
30	14.99012	151.5028	46.1574	14.20681	195.8411	66.7238
31	14.87498	158.0203	48.9566	14.05976	204.1648	70.9706
32	14.75491	164.8162	51.9595	13.90712	212.8047	75.5028
33	14.62981	171.8977	55.1785	13.74882	221.7652	80.3352
34	14.44953	179.2716	58.6264	13.58478	231.0501	85.4824
35	14.36398	186.9444	62.3164	13.41497	240.6623	90.9593
36	14.22304	194.9221	66.2622	13.23933	250.6040	96.7805
37	14.07662	203.2104	70.4777	13.05785	260.8765	102.9610
38	13.92461	211.8144	74.9770	12.87052	271.4799	109.5154
39	13.76695	220.7386	79.7749	12.67736	282.4136	116.4579

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $({}^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $({}^2A_{x:x+10})$
40	13.60357	229.9867	84.8858	12.47840	293.6755	123.8024
41	13.43441	239.5619	90.3247	12.27370	305.2625	131.5623
42	13.25943	249.4664	96.1064	12.06333	317.1700	139.7502
43	13.07861	259.7015	102.2457	11.84740	329.3924	148.3778
44	12.89194	270.2677	108.7571	11.62604	341.9222	157.4559
45	12.69943	281.1642	115.6552	11.39940	354.7507	166.9939
46	12.50112	292.3892	122.9537	11.16767	367.8678	177.0001
47	12.29706	303.9398	130.6661	10.93105	381.2615	187.4810
48	12.08733	315.8114	138.8051	10.68978	394.9184	198.4414
49	11.87202	327.9986	147.3826	10.44412	408.8233	209.8841
50	11.65127	340.4941	156.4093	10.19438	422.9597	221.8099
51	11.42522	353.2895	165.8951	9.94087	437.3092	234.2171
52	11.19405	366.3746	175.8482	9.68395	451.8518	247.1016
53	10.95797	379.7377	186.2752	9.42400	466.5661	260.4567
54	10.71721	393.3656	197.1814	9.16142	481.4292	274.2728
55	10.47203	407.2435	208.5696	8.89664	496.4168	288.5375
56	10.22273	421.3546	220.4410	8.63011	511.5030	303.2353
57	9.96964	435.6810	232.7940	8.36232	526.6612	318.3475
58	9.71308	450.2029	245.6250	8.09375	541.8633	333.8526
59	9.45345	464.8990	258.9275	7.82491	557.0805	349.7258
60	9.19114	479.7465	272.6922	7.55633	572.2833	365.9390
61	8.92659	494.7213	286.9070	7.28853	587.4417	382.4614
62	8.66024	509.7977	301.5568	7.02206	602.5251	399.2593
63	8.39257	524.9491	316.6234	6.75745	617.5030	416.2961
64	8.12406	540.1477	332.0853	6.49524	632.3449	433.5327
65	7.85522	555.3647	347.9183	6.23597	647.0206	450.9279
66	7.58658	570.5707	364.0947	5.98016	661.5006	468.4383
67	7.31867	585.7356	380.5839	5.72831	675.7560	486.0192
68	7.05202	600.8289	397.3525	5.48092	689.7590	503.6243
69	6.78718	615.8203	414.3642	5.23847	703.4830	521.2065
70	6.52467	630.6790	431.5803	5.00138	716.9030	538.7185
71	6.26504	645.3750	448.9598	4.77008	729.9954	556.1128
72	6.00881	659.8785	466.4595	4.54495	742.7386	573.3422
73	5.75650	674.1606	484.0346	4.32634	755.1127	590.3606
74	5.50858	688.1934	501.6393	4.11456	767.1002	607.1233
75	5.26555	701.9503	519.2266	3.90989	778.6857	623.5869
76	5.02783	715.4057	536.7489	3.71254	789.8559	639.7107
77	4.79586	728.5362	554.1588	3.52273	800.6001	655.4561
78	4.57002	741.3197	571.4091	3.34060	810.9096	670.7874
79	4.35066	753.7364	588.4536	3.16625	820.7782	685.6720

Illustrative Life Table: Joint Life Actuarial Functions, $i = 0.06$

Age	$\ddot{a}_{xx}$	1,000 A_{xx}	1,000 $({}^2A_{xx})$	$\ddot{a}_{x:x+10}$	1,000 $A_{x:x+10}$	1,000 $({}^2A_{x:x+10})$
80	4.13809	765.7683	605.2473	2.99977	830.2020	700.0806
81	3.93260	777.3999	621.7467	2.84117	839.1791	713.9874
82	3.73442	788.6175	637.9108	2.69046	847.7098	727.3701
83	3.54375	799.4102	653.7007	2.54760	855.7965	740.2101
84	3.36075	809.7690	669.0804	2.41251	863.4431	752.4921
85	3.18552	819.6876	684.0169	2.28509	870.6554	764.2049
86	3.01814	829.1617	698.4806	2.16521	877.4407	775.3401
87	2.85866	838.1892	712.4451	2.05273	883.8075	785.8931
88	2.70706	846.7701	725.8879	1.94748	889.7655	795.8619
89	2.56332	854.9067	738.7899	1.84925	895.3253	805.2478
90	2.42735	862.6027	751.1355	1.75786	900.4984	814.0543
91	2.29908	869.8636	762.9129	1.67309	905.2969	822.2875
92	2.17836	876.6967	774.1136	1.59471	909.7331	829.9554
93	2.06505	883.1102	784.7323	1.52251	913.8199	837.0680
94	1.95899	889.1137	794.7670	1.45626	917.5703	843.6367
95	1.85998	894.7179	804.2185	1.39571	920.9973	849.6744
96	1.76783	899.9341	813.0901	1.34065	924.1140	855.1951
97	1.68232	904.7742	821.3876	1.29084	926.9335	860.2140
98	1.60324	909.2506	829.1188	1.24605	929.4689	864.7475
99	1.53035	913.3762	836.2934	1.20604	931.7333	868.8126
100	1.46344	917.1638	842.9228	1.17060	933.7399	872.4279
101	1.40226	920.6266	849.0197	1.13946	935.5020	875.6129
102	1.34659	923.7777	854.5980	1.11241	937.0336	878.3888
103	1.29620	926.6301	859.6727	1.08917	938.3489	880.7785
104	1.25086	929.1969	864.2600	1.06949	939.4630	882.8066
105	1.21032	931.4911	868.3771	1.05308	940.3917	884.5002
106	1.17437	933.5261	872.0421	1.03965	941.1518	885.8881
107	1.14277	935.3151	875.2746	1.02889	941.7609	887.0017
108	1.11526	936.8720	878.0956	1.02047	942.2374	887.8735
109	1.09161	938.2110	880.5276	1.01406	942.6001	888.5376
110	1.07154	939.3470	882.5952	1.00934	942.8678	889.0280

Appendix 2B

ILLUSTRATIVE SERVICE TABLE

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	s_x
30	100 000	100	19 900	—	—	1.00
31	79 910	80	14 376	—	—	1.06
32	65 454	72	9 858	—	—	1.13
33	55 524	61	5 702	—	—	1.20
34	49 761	60	3 971	—	—	1.28
35	45 730	64	2 693	46	—	1.36
36	42 927	64	1 927	43	—	1.44
37	40 893	65	1 431	45	—	1.54
38	39 352	71	1 181	47	—	1.63
39	38 053	72	989	49	—	1.74
40	36 943	78	813	52	—	1.85
41	36 000	83	720	54	—	1.96
42	35 143	91	633	56	—	2.09
43	34 363	96	550	58	—	2.22
44	33 659	104	505	61	—	2.36
45	32 989	112	462	66	—	2.51
46	32 349	123	421	71	—	2.67
47	31 734	133	413	79	—	2.84
48	31 109	143	373	87	—	3.02
49	30 506	156	336	95	—	3.21
50	29 919	168	299	102	—	3.41
51	29 350	182	293	112	—	3.63
52	28 763	198	259	121	—	3.86
53	28 185	209	251	132	—	4.10
54	27 593	226	218	143	—	4.35
55	27 006	240	213	157	—	4.62
56	26 396	259	182	169	—	4.91
57	25 786	276	178	183	—	5.21
58	25 149	297	148	199	—	5.53
59	24 505	316	120	213	—	5.86

Age x	$l_x^{(\tau)}$	$d_x^{(d)}$	$d_x^{(w)}$	$d_x^{(i)}$	$d_x^{(r)}$	S_x
60	23 856	313	—	—	3 552	6.21
61	19 991	298	—	—	1 587	6.56
62	18 106	284	—	—	2 692	6.93
63	15 130	271	—	—	1 350	7.31
64	13 509	257	—	—	2 006	7.70
65	11 246	204	—	—	4 448	8.08
66	6 594	147	—	—	1 302	8.48
67	5 145	119	—	—	1 522	8.91
68	3 504	83	—	—	1 381	9.35
69	2 040	49	—	—	1 004	9.82
70	987	17	—	—	970	10.31

Appendix 3

SYMBOL INDEX

Symbol	Page	Symbol	Page
a	608	${}^{2}\ddot{a}_{xy:\overline{n}\vert}$	281
$a(x)$	72	$\bar{a}_{x\vert y}$	286
a_x	146	$\bar{a}_{\overline{x_1x_2x_3}}$	561
$a_{\overline{K}\vert}$	146	$(aA)(x)$	622
$\bar{a}_{\overline{n}\vert}$	137	$(aA)_t$	615
$\bar{a}_{P_t}$	625	$(aC)_t$	624
$\bar{a}_{\overline{T}\vert}$	134	$(aF)_t$	624
$\bar{a}_x$	135	$(aU)_t$	624
$\bar{a}_{W_t}$	627	$(aV)(x)$	622
$\ddot{a}_x$	143	$(aV)_t$	618
$\bar{a}_r^h$	609	$A(h)$	481
$\bar{a}_{x+t}^i$	351	A_t	629
$\bar{a}_{x+t}^r$	351	A_x	111
$\ddot{a}_{\overline{K+1}\vert}$	143	$\bar{A}_x$	96
$\ddot{a}_x^{(m)}$	149	$A_x^{(m)}$	121
$\overset{\circ}{a}_x^{(m)}$	156	$\bar{A}_x^{PR}$	192
$\ddot{a}_x^{\{m\}}$	155	$A_{x:\overline{n}\vert}^{1}$	95
${}_j\ddot{a}_x$	638	$A_{x:\overline{n}\vert}$	115
${}_*\ddot{a}_x$	641	$\bar{A}_{x:\overline{n}\vert}$	102
$a_{x:\overline{n}\vert}$	147	$A_{x:\overline{n}\vert}^{\ 1}$	101
$\bar{a}_{x:\overline{n}\vert}$	137	${}_jA_x$	638
$\ddot{a}_{x:\overline{n}\vert}$	144	${}_*A_x$	641
$\ddot{a}_{x:\overline{n}\vert}^{(m)}$	153	$\bar{A}_{x:\overline{n}\vert}^{1}$	95
$\overset{\circ}{a}_{x:\overline{n}\vert}^{(m)}$	156	$\bar{A}_{x:\overline{n}\vert}^{1}$	195
$\ddot{a}_{x:\overline{n}\vert}^{\{m\}}$	155	${}^{2}A_{x:\overline{n}\vert}^{\ 1}$	101
$\bar{a}_{\overline{x:\overline{n}\vert}}$	139	${}^{2}\bar{A}_{x:\overline{n}\vert}^{1}$	96
${}^{2}\bar{a}_{x:\overline{n}\vert}$	138	${}_{m\vert}\bar{A}_x$	103
${}_{n\vert}a_x$	148	${}_{m\vert n}\bar{A}_x$	109
${}_{n\vert}\bar{a}_x$	138	A_{xy}	281
${}_{n\vert}\ddot{a}_x$	145	$A_{\overline{xy}}$	280
${}_{n\vert}\ddot{a}_x^{(m)}$	153	$A_{xy}^{(m)}$	290
$\bar{a}_{xy\vert z}^{1}$	572	$\bar{A}_{xy}^{2}$	294
$\ddot{a}_{xy}^{(m)}$	290	$\bar{A}_{xy}^{1}$	293
$\ddot{a}_{\quad}$	280		

Symbol	Page
$A_{xy:\overline{n}\rceil}$	280
$\bar{A}^{1}_{xy:\overline{n}\rceil}$	283
${}^{2}A_{xy:\overline{n}\rceil}$	281
$\bar{A}^{2}_{xy}$	565
$\bar{A}_{\overline{x_1x_2x_3}}$	561
${}_{k}AS$	486
${}_{k}\hat{AS}$	512
$(AS)_{x+h}$	351
(AAI)	526
$b(u)$	587
b_j	230
b_t	94
$b_f(t)$	600
B_t	611
$\hat{B}_{x+k}$	344
$B^{(3)}_{x+t}$	549
$B^{(j)}_{x+t}$	342
${}_{h}BP$	509
c	410
c_k	486
$\hat{c}_k$	512
$c(t)$	399
C_1	651
C_2	651
C_3	651
C_h	234
${}_{k}CV$	486
$d^{(j)}_x$	316
${}_{n}d_x$	59
${}_{n}d^{(j)}_x$	316
${}_{n}d^{(\tau)}_x$	316
${}_{t}D_j$	557
${}_{k+1}D$	513
$(DA)^{1}_{x:\overline{n}\rceil}$	117
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$\mathring{e}_{x:\overline{n}\rceil}$	71
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$e_{\overline{xy}}$	272
$\mathring{e}_{xy}$	272
$\mathring{e}_{\overline{xy}}$	272
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E	501
E_0	501
${}_{n}E_x$	101
$(ES)_{x+h+t}$	351
$ELRA$	525
f	609
$f(u;t)$	490
$f_S(s)$	34
$F_X(x)$	28
F_t	629
$F^{(k)}$	35
$F_S(s)$	34
${}_{k}F$	513
G	4, 449, 467
$\hat{G}$	531
$G(b)$	407
$G(x:\alpha,\beta)$	387
$h(x)$	453, 609
$H(r)$	600
$H(x:\alpha,\beta,x_0)$	387
${}_{u}(hp)^{(\tau)}_{x+t}$	548
$(h\mu)^{(j)}_{x+t}(u)$	548
i'_{k+1}	539
$\hat{i}_{k+1}$	512
$i(s,s+t)$	656
I_k	644
I_d	445
$I_d(x)$	17
${}_{j}i_k$	638
$(IA)_x$	115
$(I\bar{A})_x$	106
$(\bar{I}\bar{A})_x$	106

Symbol	Page
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$(IA)^1_{x:\overline{n}\rceil}$	118
J	149, 308
$j(s,\ s+t,\ s+u)$	656
${}_t\bar{k}_x$	214
K	55
$K(x)$	54
$K(xy)$	267
$K(\overline{xy})$	271
l_x	58, 593
$l_{[x]+k}$	75
$l_x^{(\tau)}$	316
$l(x,u)$	587
$l_f(x,u)$	600
L	170, 417
L_1	416
L_x	70
$L(h)$	481
${}_tL$	206
${}_tL^2$	489
${}_tL_e$	467
${}_tL_e^2$	489
$\mathfrak{L}(x)$	58
$\mathfrak{L}_x^{(\tau)}$	316
$m(x)$	68, 614
m_x	70
$m_x^{(j)}$	322
$m_x^{(\tau)}$	321
$m_x'^{(j)}$	322
$\mathrm{M}_x(t)$	11
$M(x)$	614
$n(u)$	608
N	367
$N(t)$	406, 519
$p(j)$	641
$p(x)$	369
p_k	368
$p_{[x]+r}$	75
$p^{*n}(x)$	370
${}_tp_x$	53

Symbol	Page
${}_tp_x^{(\tau)}$	310
${}_tp_x'^{(j)}$	319
${}_tp_{\lambda y}$	264
${}_tp_{\overline{xy}}$	268
${}_up_{\overline{xy}+t}$	575
${}_tp_{\overline{x_1x_2x_3}}^{\ k}$	560
$P(x)$	318, 622
$P(s,t)$	656
P_t	616
${}^TP_t$	610
P^a	501
P_x	180
${}_jP_x$	638
${}_*P_x$	638
$P^A_{x:\overline{n}\rceil}$	473
$P_{x:\overline{n}\rceil}$	183
$P_{\overline{xy}}$	573
$P^1_{x:\overline{n}\rceil}$	183
$P_{x:\overline{n}\rceil}^{\ 1}$	183
$\tilde{P}^1_{x:\overline{n}\rceil}$	195
$P^{*n}(x)$	369
${}_hP_x$	183
${}_hP_{x:\overline{n}\rceil}$	183
$(Pa)(x)$	622
$(Pa)_t$	619
$P(\bar{A}_{x:\overline{n}\rceil})$	188
$P({}_{n\rceil}\ddot{a}_x)$	183
$\bar{P}({}_{n\rceil}\bar{a}_x)$	173
$\tilde{P}(\bar{A}_x)$	170
$P^{(m)}(\bar{A}_x)$	189
$P^{\{m\}}(\bar{A}_x)$	191
$P(\bar{A}_x^{PR})$	192
$\tilde{P}(\bar{A}_{x:\overline{n}\rceil})$	173
$\tilde{P}(\bar{A}^1_{x:\overline{n}\rceil})$	173
$\tilde{P}(A_{x:\overline{n}\rceil}^{\ 1})$	173
$P^{(m)}(\bar{A}_{x:\overline{n}\rceil})$	189
$P^{(m)}(\bar{A}^1_{x:\overline{n}\rceil})$	189
${}_h\tilde{P}(\bar{A}_x)$	173
${}_h\bar{P}(\bar{A}_{x:\overline{n}\rceil})$	173
${}_hP^{(m)}(\bar{A}_x)$	189
${}_hP^{(m)}(\bar{A}_{x:\overline{n}\rceil})$	189
${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}\rceil})$	191
$P(\bar{A}^2_{\overline{xyz}})$	573
$P(\bar{A}^{\ 2}_{\underset{1}{xyz}})$	573

Symbol	Page
$\tilde{P}^{1}_{x:\overline{n}\|}$	195
$q_{[x]+r}$	79
$q_x^{(d)}$	350
$q_x^{(i)}$	350
$q_x^{(r)}$	350
$q_x^{(w)}$	350
$\hat{q}^{(j)}_{x+k}$	512
q_{xy}	263
${}_{k\|}q_x$	54
${}_tq_x$	53
${}_tq_x^{(j)}$	310
${}_tq_x^{(\tau)}$	310
${}_tq_x^{\prime(j)}$	319
${}_{t\|u}q_x$	53
${}_nq^{1}_{xy}$	291
${}_nq^{2}_{xy}$	292
${}_{k\|}q_{xy}$	267
${}_nq^{2}_{1\,xyz}$	566
${}_\infty q^{3}_{12\,wxyz}$	569
r	608
r_C	526
r_F	525
r_N	527
$(rA)_t$	611
$(rF)_t$	629
$(rV)_t$	629
R	410, 594, 601
$\tilde{R}$	401
$R(x,h,t)$	352
$s(x)$	52
$\ddot{s}_{\overline{n}\|}$	194
$s(x,u)$	587
$\bar{s}_{x:\overline{n}\|}$	140
$\ddot{s}_{x:\overline{n}\|}$	146
S	27, 367
$S(t)$	399
S_n	401
S_y	351
${}_kSC$	500
T	55, 400
$\tilde{T}$	401
$T(x)$	52
T_x	70, 347, 593
$T(xy)$	263
$T(\overline{xy})$	268
$u(w)$	4
$U(h)$	481
$U(t)$	399, 481
U_t	629
U_n	401
$\hat{U}_n$	405
v_t	94
$\tilde{v}_n$	645
V_i	406
V_t	629
${}_kV_x$	215
${}_kV_{x:\overline{n}\|}$	216
${}_kV^{1}_{x:\overline{n}\|}$	216
${}_kV^{\underline{1}}_{x:\overline{n}\|}$	216
${}_kV^{FPT}_x$	521
${}_tV^{\underline{1}}_{\overline{(xy)}:\overline{n}\|}$	574
${}^h_kV_x$	216
${}^h_kV_{x:\overline{n}\|}$	216
${}^h_kV^{(m)}_{x:\overline{n}\|}$	222
${}^h_kV^{Mod}_{x:\overline{n}\|}$	517
${}_kV(_{n\|}\ddot{a}_x)$	216
${}_t\bar{V}(_{n\|}\bar{a}_x)$	212
${}_t\bar{V}(\bar{A}_x)$	206
${}_t\bar{V}(\bar{A}_{x:\overline{n}\|})$	212
${}_t\bar{V}(\bar{A}^{1}_{x:\overline{n}\|})$	212
${}_t\bar{V}(A^{\underline{1}}_{x:\overline{n}\|})$	212
${}_t\bar{V}(\bar{A}_x)^{Mod}$	518
${}_tV(\bar{A}_{\overline{xy}})$	574
${}_kV^{\{1\}}(\bar{A}_x)$	224
${}_kV(\bar{A}^{PR}_x)$	225
${}^h_kV(\bar{A}^{1}_{x:\overline{n}\|})$	221
${}^h_t\bar{V}(\bar{A}_x)$	212
${}^h_t\bar{V}(\bar{A}_{x:\overline{n}\|})$	212
${}^h_kV^{\{m\}}(\bar{A}_{x:\overline{n}\|})$	224
$w(x)$	608
W_i	354, 402
W_t	608
${}_kW$	503

Symbol	Page
${}_kW_x$	503
${}_kW_{x:\overline{n}\rceil}$	503
${}^{h}_{k}W_x$	503
$(Wa)_t$	627
${}_k\bar{W}(\bar{A}_x)$	503
${}_k\bar{W}(\bar{A}_{x:\overline{n}\rceil})$	503
${}^{h}_{k}\bar{W}(\bar{A}_x)$	503
(x)	52
$(x_1x_2\cdots x_m)$	263
$(\overline{x_1x_2\cdots x_m})$	268
$\overline{x_1x_2\cdots x_m}^{\,k}$	556
$\overline{x_1x_2\cdots x_m}^{\,[k]}$	556
X_i	27, 367
$X(\theta)$	617
Y	134
$y(s,s+m)$	657
$Y(t,n)$	659
z_t	94
Z	94
${}_mZ_y$	352
α	294, 519
$\alpha(m)$	152
$\bar{\alpha}$	520
α^{CRVM}	522
β	519, 610
$\beta(m)$	152
$\bar{\beta}$	520
β^{CRVM}	522
$\beta(x,u)$	600
$\Gamma(\alpha)$	374
δ	96
δ_t	96
θ	41, 617
$\lambda(t)$	625
$\lambda(t,n)$	659
Λ	373
Λ_h	242
$\mu(x)$	55
$\mu_x(t)$	79
$\mu_x^{(d)}$	351
$\mu_x^{(i)}$	351
$\mu_x^{(w)}$	351
$\mu_x^{(j)}(t)$	311
$\mu_x^{(\tau)}(t)$	311
$\mu_{xy}(t)$	266
$\mu_{\overline{xy}}(t)$	270
$\mu(x,u)$	589
π_h	230
π_t	597
ρ	610
τ	310, 608
$\phi(x)$	600
$\phi(x,u)$	600
$\psi(u)$	400
$\tilde{\psi}(u)$	401
$\psi(u,t)$	400
$\psi(u;w)$	427
$\tilde{\psi}(u,w)$	404
ω	63

Appendix 4

GENERAL RULES FOR SYMBOLS OF ACTUARIAL FUNCTIONS

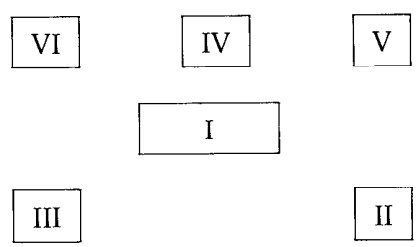
An actuarial function is represented by a principal symbol and a combination of auxiliary symbols such as letters, numerals, double dots, circles, hats, and horizontal and vertical bars. The principal symbol expresses the general definition of the function; choice and placement of the auxiliary symbols at the top and corners give precise meaning. We review the rules for selecting and placing the symbols and show one or more functional forms in common application areas.

This notation is based upon the system of International Actuarial Notation (IAN) that was originally adopted by the Second International Congress of Actuaries in London in 1898 and is modified periodically under the guidance of the Permanent Committee of Actuarial Notations of the International Actuarial Association. IAN is a basic system of principles that does not cover all areas of actuarial applications. In this text these principles have been followed, and sometimes extended, to construct consistent notation where needed.

This Appendix is meant to provide the reader with an overview of basic patterns for expressing the symbols appearing in this book. Although it is a good introduction to IAN, it is not exhaustive. Authoritative sources for further reference are:

- Actuarial Society of America, "International Actuarial Notation," *Transactions*, XLVIII, 1947: 166–176.
- Faculty of Actuaries, *Transactions*, XIX, 1950: 89.
- *Journal of the Institute of Actuaries*, LXXV, 1949: 121.

An actuarial symbol can be viewed as illustrated below. Box I represents the principal symbol, the others subscripts or superscripts. The roman numerals in the boxes correspond to the section designations of this Appendix.



Section I. Center

Principal Symbol	*Description*	*Topic*
i	Effective rate of interest for a time period, usually 1 year, or with a superscript in Position V, a nominal rate.	Interest
v	Present value of 1 due at the end of the effective interest period, usually 1 year.	
δ	Force of interest, usually stated as an annual rate.	
d	Effective rate of interest-in-advance, or discount rate, for a time period of usually 1 year, or with a superscript in Position V, a nominal rate. This symbol never has a subscript in Position II.	
l	Expected number, or number, of survivors at a given age.	Life Tables
d	Expected number, or number, of those dying within a given time period. This symbol always has a subscript in Position II.	
p	Probability of surviving for a given time period.	
q	Probability of dying within a given time period.	
μ	Force of mortality, usually stated on an annual basis.	
m	Central death rate for a given time period.	
L	Expected number, or number, of years lived within a time period by the survivors at the beginning of the period.	
T	Expected total, or total, future lifetime of the survivors at a given age. (The above are survivorship group definitions of the life table functions denoted by l, d, L, and T. For the alternative stationary population definitions, see Chapter 19.)	
A	Actuarial present value (net single premium) of an insurance or pure endowment of 1.	Life Insurance and Pure Endowments

Principal Symbol	*Description*	*Topic*
(IA)	Actuarial present value (net single premium) of an insurance with a benefit amount of 1 at the end of the first year, increasing linearly at a rate of 1 per year.	
(DA)	Actuarial present value (net single premium) of a term insurance with an initial benefit amount equal to the term and decreasing linearly at the rate of 1 per year.	
E	Actuarial present value of a pure endowment of 1.	
a	Actuarial present value of an annuity of 1 per time period, usually 1 year.	Annuities
s	Actuarial accumulated value of an annuity of 1 per time period, usually 1 year.	
(Ia)	Actuarial present value of an annuity payable at the rate of 1 per year at the end of the first year and increasing linearly at a rate of 1 per year.	
(Da)	Actuarial present value of a temporary annuity with an initial payment rate equal to the term and decreasing linearly at a rate of 1 per period.	
P	Level annual premium rate to cover only benefits, usually determined by the equivalence principle.	Premiums
V	Reserve to cover future benefits in excess of future benefit premiums.	Reserves
W	Face amount of a paid-up policy purchased with a cash value equal to the reserve.	
	(Principle symbols for benefit premiums, reserves, and amounts of reduced paid-up insurance, P, V, and W, are combined with benefit symbols unless the benefit is a level unit insurance payable at the end of the year of death.)	[Examples: $\bar{P}_x$; $P(\bar{A}_x)$; ${}_{10}V^{(4)}\ (\bar{A}_{x:\overline{n}|})$; $P^{(12)}({}_{30|}\ddot{a}_{35}^{(12)})$]
S	Salary scale function used to project salaries.	Pensions
Z	Average of a given number of salary scale function values, usually at unit intervals in the independent variable.	

Section II. Lower Space to the Right

Auxiliary Symbol	*Description*	*Examples*
x; 10	A single letter or numeral is the individual's age at the commencement of the overall time period implied by the principal symbol.	a_x; $\bar{a}_{10}$ q_x; ${}_5q_{10}$ $\bar{A}_x$; A_{10}
$\overline{n}\rvert$; $\overline{10}\rvert$	A term certain is indicated by a single letter or numeral under an angle.	$A_{x:\overline{n}\rvert}$; $\ddot{a}_{\overline{10}\rvert}$
$[x]$; [35] $[x] + t$; $[35 - n]$ $+ n$	Alphanumeric expressions enclosed by brackets indicate the age at which the life was selected. A term, representing duration since selection, may be added to the bracketed expression to express the attained age of the life.	$l_{[x]}$; $l_{[x]+10}$ $A_{[35]}$; $\ddot{a}^{i}_{[35-n]+n}$
xyz or $x{:}y{:}z$ $25{:}\overline{10}\rvert$	Two or more alphanumeric characters indicate a joint status that survives until the first death or expiration of the indicated lives and terms certain.	l_{xyz}; $A_{x:y:z}$ $\ddot{a}_{25:\overline{10}\rvert}$; $P_{25:\overline{10}\rvert}$
$\overset{\frown}{}$	This symbol emphasizes the joint status when ambiguity is possible.	$A_{\overset{1}{\overset{\frown}{xy}}:z}$
$\overset{1}{x}{:}\overline{10}\rvert$; $\overset{2}{\underset{1}{xyz}}$	Numerals can be placed above or below the individual statuses of a collection of alphanumeric characters to show the order in which the units are to fail for an (insurable) event to occur. Benefits are payable upon the failure of the status with a numeral above it.	$\bar{A}^{1}_{x:\overline{n}\rvert}$; ${}_{\infty}q^{3}_{\underset{12}{xyz}}$
$\overline{xyz}$; $\overline{65{:}60{:}\overline{10}\rvert}$	A horizontal bar over a collection of alphanumeric characters defines a status that survives until the last survivor of the individual statuses fails.	$a_{\overline{xyz}}$; $\bar{A}_{\overline{xy:\overline{n}\rvert}}$
$\underset{xyz}{\underline{r}}$; $\underset{x:y:10}{\underline{[r]}}$	A single alphanumeric character, say, r, above the right end of the bar over the set of alphanumeric characters defines a status that survives as long as at least r of the individual statuses survive. If the r is enclosed in brackets, the status exists only while exactly r of the individual statuses survive.	$\bar{a}_{\underset{xyz}{\underline{[2]}}}$; $\bar{A}_{\underset{xyz}{\underline{2}}}$

Auxiliary Symbol	*Description*	*Examples*
$y\|x$; 60|55 $\overset{1}{yz}\|x$	A vertical bar separating the alphanumeric characters indicates that the income or coverage of the principal symbol commences upon the failure, as specified, of the status before the bar and continues until the failure of the status following the bar, providing the statuses fail in that order.	$a_{y\|x}$ $a_{\overset{1}{wyz}\|x}^{3}$

Section III. Lower Space to the Left

Auxiliary Symbol	*Description*	*Examples*
n; 15	A single alphanumeric character shows the time for which the principal symbol is evaluated. For an annual premium, P, this position shows the maximum number of years for which the premiums are paid if this is less than the period of coverage of an insurance or the period of deferral for a deferred annuity.	${}_{n}p_x$; ${}_{15}E_{30}$ ${}_{20}P_{25}$; ${}_{20}V_{40:\overline{30}\|}$
$n\|m$; $n\|$	An alphanumeric pair separated by a vertical bar indicates a period of deferment (left of the bar) and a period following deferment (right of the bar). In some cases, when either is equal to 1 or infinity, it can be omitted.	${}_{n\|m}q_x$; ${}_{n\|}\bar{a}_x$

Section IV. Top Center

Auxiliary Symbol	*Description*	*Examples*
¨	The double dot (dieresis) on an annuity symbol indicates that the payments are at the beginning of the periods, that is, an annuity-due. Without the dieresis, the annuity is an annuity-immediate with payments at the ends of the periods.	$\ddot{a}_x$; $\ddot{s}_{\overline{40}\rceil}$
¯	A horizontal bar indicates that the frequency of events is infinite. For annuities the payments are considered to be made continuously, and for insurances the benefit is paid at the moment of failure.	$\bar{a}_x$; $\bar{A}_x$ $_3\bar{V}_x$; $\bar{P}(\bar{A}_x)$
°	A circle (degree sign) means that the benefit or lifetime is complete, that is, credited up to the time of death.	$\mathring{a}_x$; $\mathring{e}_x$

Section V. Upper Space to the Right

Auxiliary Symbol	*Description*	*Examples*
(m); (12)	An alphanumeric character in parentheses shows the number of annuity payments in an interest period, usually 1 year. For an insurance it is the number of periods in a year at the end of which the death benefit can be paid. On multiple decrement symbols it indicates the cause of decrement to be used or that the total of all decrements is to be used.	$s^{(12)}_{\overline{10}\rceil}$; $A^{(m)}_x$ $q^{(2)}_x$; $_tp^{(\tau)}_x$
$\{m\}$; {12}	An alphanumeric character in braces shows the number of apportionable annuity-due payments in a time period, usually 1 year. On a principal symbol of a premium or a reserve, it shows that premiums are paid on this basis.	$\ddot{a}^{\{12\}}_{30:\overline{20}\rceil}$; $P^{\{1\}}_{30}$ $_tV^{\{2\}}(\bar{A}_x)$
r; i	An alphabetic character indicates the special basis used for the actuarial present value.	$\ddot{a}^{r}_{65}$; $\ddot{a}^{i}_{\{x\}}$

Section VI. Upper Space to the Left

Auxiliary Symbol	*Description*	*Examples*
h; 2	Alphanumeric character indicating the number of years during which premiums are paid if this is less than the coverage period of the insurance or the deferral period of the deferred annuity. This is used only on the principal symbols V or W where Position III is used for the time for which the function is evaluated.	${}^{h}_{5}V_{30}$
	In this text a new use for this position is to show that the actuarial present value of an annuity or an insurance is calculated at a multiple of the assumed force of interest.	${}^{2}\bar{A}_{x}$; ${}^{2}\ddot{a}_{20:\overline{10}\rceil}$

Appendix 5

SOME MATHEMATICAL FORMULAS USEFUL IN ACTUARIAL MATHEMATICS

The purpose here is not to recall familiar standard formulas and techniques, but to indicate some that may be less familiar to actuarial students.

Calculus

If

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t)\, dx,$$

then

$$\frac{dF(t)}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x, t)\, dx + f(\beta(t), t)\, \frac{d}{dt}\, \beta(t) - f(\alpha(t), t)\, \frac{d}{dt}\, \alpha(t).$$

Calculus of Finite Differences

Operators

a. Shift:

$$\mathrm{E}[f(x)] = f(x + 1)$$

b. Difference:

$$\Delta f(x) = f(x + 1) - f(x) = (\mathrm{E} - 1) f(x)$$

c. Repeated differences:

$$\begin{aligned} \Delta^n f(x) &= \Delta[\Delta^{n-1} f(x)] \\ &= (\mathrm{E} - 1)^n f(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + k) \end{aligned}$$

d. Difference of a product:

$$\Delta[f(x)\, g(x)] = f(x + 1)\, \Delta g(x) + g(x)\, \Delta f(x)$$

e. Antidifference:

If

$$\Delta f(x) = g(x),$$

then

$$\Delta^{-1} g(x) = f(x) + w(x)$$

where

$$w(x) = w(x + 1).$$

Applications

a. Representation of a polynomial (Newton's formula): Let $p_n(x)$ be a polynomial of degree n; then

$$p_n(x) = \sum_{k=0}^{n} \binom{x - a}{k} \Delta^k p_n(a).$$

b. Summation of series:

If

$$\Delta F(x) = f(x),$$

then

$$\begin{aligned} f(1) &= F(2) - F(1) \\ f(2) &= F(3) - F(2) \\ &\vdots \\ f(n) &= F(n + 1) - F(n) \end{aligned}$$

$$\sum_{x=1}^{n} f(x) = F(n + 1) - F(1) = \Delta^{-1} f(x)\Big|_1^{n+1}.$$

c. Summation by parts:

$$\sum_{x=1}^{n} g(x)\, \Delta f(x) = f(x)\, g(x)\Big|_1^{n+1} - \Delta^{-1}[f(x + 1)\, \Delta g(x)]\Big|_1^{n+1}$$

[Proof: Sum each side of the equation for $\Delta[f(x)g(x)]$ from $x = 1$ to $x = n$.]

Probability Distributions

Discrete Distributions	p.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments	
				Mean	Variance
Binomial	$\binom{n}{x}p^x q^{n-x}$, $x = 0, 1, \ldots n$	$0 < p < 1$ $q = 1 - p$	$(pe^s + q)^n$	np	npq
Bernoulli	Special case $n = 1$				
Negative Binomial	$\binom{r+x-1}{x}p^r q^x$, $x = 0, 1, 2, \ldots$	$0 < p < 1$ $q = 1 - p$ $r > 0$	$\left(\frac{p}{1 - qe^s}\right)^r$, $qe^s < 1$	$\frac{rq}{p}$	$\frac{rq}{p^2}$
Geometric	Special case $r = 1$				
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\lambda > 0$	$e^{\lambda(e^s - 1)}$	λ	λ
Uniform	$\frac{1}{n}$, $x = 1, \ldots, n$	n, a positive integer	$\frac{e^s(1 - e^{sn})}{n(1 - e^s)}$, $s \neq 0$ 1, $s = 0$	$\frac{n + 1}{2}$	$\frac{n^2 - 1}{12}$

Continuous Distributions	p.d.f.	Restrictions on Parameters	Moment Generating Function, $M(s)$	Moments	
				Mean	Variance
Uniform	$\frac{1}{b-a}, \quad a < x < b$	—	$\frac{e^{bs} - e^{as}}{(b-a)s}, s \neq 0$ $1, s = 0$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\mu)^2/2\sigma^2]$, $-\infty < x < \infty$	$\sigma > 0$	$\exp(\mu s + \sigma^2 s^2/2)$	μ	σ^2
Gamma	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$	$\alpha > 0, \beta > 0$	$\left(\frac{\beta}{\beta - s}\right)^\alpha, s < \beta$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Exponential	Special case $\alpha = 1$				
Chi-square	Special case $\alpha = \frac{k}{2}, \beta = \frac{1}{2}$	k, a positive integer			
Inverse Gaussian	$\frac{\alpha}{\sqrt{2\pi\beta}} x^{-3/2} \exp\left[-\frac{(\beta x - \alpha)^2}{2\beta x}\right]$, $x > 0$	$\alpha > 0, \beta > 0$	$\exp\left[\alpha\left(1 - \sqrt{1 - \frac{2s}{\beta}}\right)\right]$, $s < \frac{\beta}{2}$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Pareto	$\alpha x_0^\alpha / x^{\alpha+1}, x > x_0$	$x_0 > 0, \alpha > 0$		$\frac{\alpha x_0}{\alpha - 1}$ $\alpha > 1$	$\frac{\alpha x_0^2}{(\alpha-2)(\alpha-1)^2}$ $\alpha > 2$
Lognormal	$\frac{1}{x\sigma\sqrt{2\pi}} \exp[-(\log x - m)^2/2\sigma^2]$, $x > 0$	$-\infty < m < \infty$ $\sigma > 0$		$e^{m+\sigma^2/2}$	$(e^{\sigma^2} - 1)e^{2m+\sigma^2}$

Appendix 7

ANSWERS TO EXERCISES

Chapter 1

1.1. a. and b.

w	$u(w)$	$u(w_1,w_2)$	$u(w_1,w_2,w_3)$
0	-1.00	125×10^{-6}	-48×10^{-10}
4 000	-0.500	93×10^{-6}	-34×10^{-10}
6 700	-0.250	78×10^{-6}	-14×10^{-10}
8 300	-0.125	74×10^{-6}	—
10 000	0.000	—	—

1.2. b. 2, 2 d. 2 log 2

1.3. c. Var(X)

1.7. a. Yes, for all w b. $(90 < w < 100) \cup (w > 110)$

1.11. $-\frac{n}{2\alpha} \log (1 - 2\alpha)$

1.12. a. $G = 400 \log \frac{13}{12} = 32.02$ b. $G = 150 \log \frac{3}{2} = 60.82$

1.13. a. 30 b. 26

1.14. Complete insurance

1.17. a. $-[1 - F(d)]$

1.18. a. 10, 100

1.19. a. 50, $\frac{2{,}500}{3}$ b. $k = 0.25$, $d = 50$

c. $\text{Var}[X - I_1(X)] = 468.75$, $\text{Var}[X - I_2(X)] = 260.42$

Chapter 2

2.1. $\frac{1}{2}, \frac{19}{4}$

2.2. $\frac{1}{2}, \frac{77}{12}$

2.3. $\frac{35}{4}, \frac{1{,}085}{48}$

2.4. $\frac{7}{4}, \frac{77}{48}$

2.5. $\frac{49}{4}, \frac{735}{16}$

2.6. $\frac{a}{100}$ and $a^2\left(\frac{197}{30{,}000}\right)$

2.7.

x	$F_S(x)$
0	0.2268
1	0.2916
2	0.4374
3	0.6210
4	0.7434
5	0.8586
6	0.9018
7	0.9582
8	0.9762
9	0.9918
10	0.9948
11	0.9988
12	0.9996
13	1.0000

2.8. c. $\frac{1}{48}, \frac{1}{6}, \frac{1}{2}$

2.9. $E[X] = \frac{\alpha}{\beta}$, $\text{Var}(X) = \frac{\alpha}{\beta^2}$

2.10. $E[X] = 1$, $\text{Var}(X) = \frac{1}{3}$; $E[Y] = \frac{3}{2}$, $\text{Var}(Y) = \frac{3}{4}$; $\Pr(X + Y > 4) \cong 0.0748$; $\Pr(X + Y > 4) = 0.0833$

2.11. a. $b = -1$, $c = 1$, $d = a$ or $b = 1$, $c = 0$, $d = -a$
b. 0.0228, 0.1587, 0.5000

2.12. a. 18, 36 b. 27.8713, 31.9607

2.13. a. 0.0041 b. 0.0045

2.14. 3.56; that is, 35,600

2.15. a. 6.4, 6.144 b. $7(10^4)$, $17.072(10^8)$ c. 1.37341

2.16. 0.0062

3.1.

$s(x)$	$F(x)$	$f(x)$	$\mu(x)$
$\cos x$	$1 - \cos x$	$\sin x$	—
—	$1 - e^{-x}$	e^{-x}	1
$\frac{1}{1+x}$	—	$\frac{1}{(1+x)^2}$	$\frac{1}{1+x}$

3.2. a. $\exp\left[\frac{-B}{\log c}(c^x - 1)\right]$ b. $\exp(-ux^{n+1})$ $u = \frac{k}{n+1}$

c. $\left(1 + \frac{x}{b}\right)^{-a}$

3.3. $\mu(x) = \frac{x^2}{4}$, $f(x) = \frac{x^2}{4}e^{-x^3/12}$, $F(x) = 1 - e^{-x^3/12}$

3.4. a. $\int_0^\infty \mu(x)\,dx < \infty$ b. $s'(x) > 0$ for some x including $x = 1, 2$

c. $\int_0^\infty f_X(x)\,dx = 2^n\Gamma(n) > 1$ for $n \geq 1$

3.5. a. $\frac{1}{100 - x}$ b. $\frac{x}{100}$ c. $\frac{1}{100}$ d. $\frac{3}{10}$

3.6. a. $1 - \frac{t}{60}$ b. $\frac{1}{60 - t}$ c. $\frac{1}{60}$

3.7. a. $\frac{8}{9}$ b. $\frac{1}{8}$ c. $\frac{1}{8}$ d. $\frac{1}{128}$ e. $\frac{128}{3}$

3.9. 0.001994

3.10. $f_X(x) = \binom{10}{x}(0.77107)^x(0.22893)^{10-x}$ $x = 0, 1, 2, \ldots, 10$

$E[\mathfrak{L}(x)] = 7.7107$, $\text{Var}[\mathfrak{L}(x)] = 1.765211$

3.11. a. $\frac{9}{4}$ for each b. $\frac{27}{16}$ for each c. $-\frac{1}{3}$

3.12. a. ${}_5q_0 = 0.01505$ is more than 10 times ${}_5q_5 = 0.001503$

b. ${}_{55|5}q_{25} = 0.156729$

3.15. 1,436.19

3.18. a. $\frac{1}{c}$ b. $\frac{1}{c^2}$ c. $\frac{1}{c}\log 2$ d. 0

3.19. a. $te^{-t^2/2}$ b. $\sqrt{\frac{\pi}{2}}$

3.20. a. $\frac{(100 - x)}{2}$ b. $\frac{(100 - x)^2}{12}$ c. $\frac{(100 - x)}{2}$

3.23. a. $\mathring{e}_x = \dfrac{10 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

$e_x = \dfrac{9 - x}{2} \quad x = 0, 1, 2, \ldots, 9$

3.24. a. $u(0) = e^{-\lambda}, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{\lambda}{x + 1}$

b. $u(0) = (1 - p)^n, \quad -\dfrac{c(x)}{d(x)} = 0, \quad \dfrac{1}{d(x)} = \dfrac{(n - x)p}{(x + 1)(1 - p)}$

3.25. a. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 \quad \dfrac{1}{d(x)} = v$

b. $u(0) = 0 \quad -\dfrac{c(x)}{d(x)} = 1 + i \quad \dfrac{1}{d(x)} = 1 + i$

3.28. Uniform distribution: 0.989709
Constant force: 0.989656
Balducci: 0.989602

3.29. a. 77.59 b. 29.11

3.30. a. 0.044 b. 0.04421

3.31.

	Uniform Distribution	Constant Force	Balducci
a.	0.012696	0.012616	0.012537
b.	0.013676	0.013770	0.013865
c.	0.013770	0.013770	0.013770

3.35. a. $\dfrac{\alpha}{\omega - x}$ b. $\dfrac{\omega - x}{\alpha + 1}$

3.36. a. 0.00142 b. 0.99867

3.37. a. 0.317 b. 0.140

3.39. 0.97920

3.40. $\log\left(1 - \dfrac{q_{[x]}}{2}\right) - \log(1 - q_{[x]})$

3.41. $q'_x < 2q_x$

3.43. a. $\left(\dfrac{1 + Bc^x}{1 + B}\right)^{-A/(B \log c)}$

3.44. a. $\dfrac{5^7}{4^{18}}$ b. 77.2105

3.45. b. $-\log(1 - q_x)$ c. $\dfrac{-q_x^2}{(1 - q_x)\log(1 - q_x)}$ d. $\dfrac{1}{45}$

3.49. $q^1_{40} = 0.0055547, \quad q_{40} = 0.0027812$

3.50. Check value $e_{40} = 35.367$

3.51. $e_{20} = 46.038$, $e_{40} = 28.366$, $e_{60} = 13.264$, $e_{80} = 3.889$, $e_{100} = 0.503$

3.52. Check value $\mathring{e}_{40} = 35.867$

3.53. Starting value $e_{y:\overline{0}|} = 0$, Check value $e_{25:\overline{20}|} = 19.369$

3.54. Starting value $e_{\omega:\overline{10}|} = 0$, Check value $e_{40:\overline{10}|} = 9.809$

3.55. $e_{15:\overline{25}|} = 24.610$

Chapter 4

4.5. b. $\bar{A}^1_{x:\overline{n}|} = \dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} A_{x:\overline{n}|}^{\ 1}$ c. $-\dfrac{\mu_{x+n}}{\delta + \mu_{x+n}} (A_{x:\overline{n}|}^{\ 1})^2$, where n satisfies (b)

d. $n = \dfrac{\log 2}{\mu + \delta}$, min $\text{Cov}(Z_1, Z_2) = -\dfrac{\mu}{4(\mu + \delta)}$

4.6. a. 0.237832 b. 0.416667

4.7. a. 0.092099 b. 0.055321

4.8. a. $\dfrac{20}{3(100 - x)}\left[1 - \left(\dfrac{20}{120 - x}\right)^3\right]$,

$\dfrac{20}{7(100 - x)}\left[1 - \left(\dfrac{20}{120 - x}\right)^7\right] - \left\{\dfrac{20}{3(100 - x)}\left[1 - \left(\dfrac{20}{120 - x}\right)^3\right]\right\}^2$

b. $\dfrac{20}{3(100 - x)}\left[10 - 10\left(\dfrac{20}{120 - x}\right)^2 - (100 - x)\left(\dfrac{20}{120 - x}\right)^3\right]$

4.10. a. $\dfrac{\mu}{(\mu + \delta)^2}$ b. $\mu\left[\dfrac{2}{(\mu + 2\delta)^3} - \dfrac{\mu}{(\mu + \delta)^4}\right]$

4.11. a. 0.407159 b. 5.554541

4.13. a. 0.5 b. 0.05

4.14. b. $(IA)_{x:\overline{m}|} = (IA)^1_{x:\overline{m}|} + mA_{x:\overline{m}|}^{\ 1}$

4.15. a. $v^{[k+(j+1)/m]}$ b. $A_x^{(m)} = \sum_{k=0}^{\infty} v^{k+1}\ {}_kp_x \sum_{j=0}^{m-1} {}_{j/m|1/m}q_{x+k}\ (1 + i)^{[1-(j+1)/m]}$

4.23. $A_x + A^1_{x:\overline{65-x}|}$

4.24. 4,007.85

4.26. a. $\dfrac{9{,}100}{14 - k}$

b. $1{,}000{,}000[{}^2A_{x:\overline{n}|}^{\ 1} - (A_{x:\overline{n}|}^{\ 1})^2] + (k\pi)^2[{}^2\bar{A}^1_{x:\overline{n}|} - (\bar{A}^1_{x:\overline{n}|})^2] - 2{,}000\ k\pi\ \bar{A}^1_{x:\overline{n}|}\ A_{x:\overline{n}|}^{\ 1}$
where π is the net single premium in (a)

4.27. a. 0.307215

4.34. a. $A^1_{20:\overline{20}|} = 0.01827$
${}^2A^1_{20:\overline{20}|} = 0.01143$
b. 110,933,839

4.35. b. $A_{\omega:\overline{0}|} = 0.0$

4.36. a. 3.06569 b. $(I\bar{A})_x = (\bar{A}^1_{x:\overline{1}|} + vp_x\,\bar{A}_{x+1}) + vp_x\,(I\bar{A})_{x+1}, \quad (I\bar{A})_\omega = 0$

c. $(\bar{I}\bar{A})_x = [(\bar{I}\bar{A})^1_{x:\overline{1}|} + vp_x\,\bar{A}_{x+1}] + vp_x\,(\bar{I}\bar{A})_{x+1}, \quad (\bar{I}\bar{A})_\omega = 0$

d. $(I\bar{A})_x = (d/\delta)[q_x + p_x\,\bar{A}_{x+1}] + vp_x\,(I\bar{A})_{x+1}$,

$$(\bar{I}\bar{A})_x = \frac{d}{\delta}\left[\left(\frac{i-\delta}{i\delta}\right)q_x + p_x\,\bar{A}_{x+1}\right] + vp_x\,(\bar{I}\bar{A})_{x+1}$$

4.38. b. $\bar{A}_{y:\overline{0}|} = 1$ c. $\bar{A}_{45:\overline{65-45}|} = 0.34743$

4.39. mean = 38,056.82, variance = 42,337,224.63

Chapter 5

5.1. a. 16.008, 12.761, 5.397 b. 3.137, 10.230, 9.523

5.2. a. 0.111, 0.251, 0.572 b. 0.0251

5.4. $-\text{Var}(v^T) = -({}^2\bar{A}_x - \bar{A}_x^2)$

5.6. a. $F_y(y) = \begin{cases} 1-(1-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$

b. $F_y(y) = \begin{cases} 1-(1-\delta y)^{\mu/\delta} & 0 \le y < \bar{a}_{\overline{n}|} \\ 1 & \bar{a}_{\overline{n}|} \le y \end{cases}$

c. $F_y(y) = \begin{cases} 1-(v^n-\delta y)^{\mu/\delta} & 0 \le y < \dfrac{v^n}{\delta} \\ 1 & \dfrac{v^n}{\delta} \le y \end{cases}$

d. $F_y(y) = \begin{cases} 0 & 0 < y \le \bar{a}_{\overline{n}|} \\ 1-(1-\delta y)^{\mu/\delta} & \bar{a}_{\overline{n}|} < y < \dfrac{1}{\delta} \\ 1 & \dfrac{1}{\delta} \le y \end{cases}$

5.7. $\bar{a}_{x:\overline{n}|} = \bar{a}_{x:\overline{1}|} - v^n\,{}_np_x\,\bar{a}_{x+n:\overline{1}|} + vp_x\,\bar{a}_{x+1:\overline{n}|} \quad \begin{matrix} n = 1, 2, \ldots \\ x = 0, 1, \ldots \end{matrix}$

$\bar{a}_{\omega:\overline{n}|} = \bar{a}_{\omega:\overline{1}|}$ which equals 1/2 by the trapezoidal rule

5.8. ${}_{n|}\bar{a}_x = v^n\,{}_np_x\,\bar{a}_{x+n:\overline{1}|} + vp_x\,{}_{n|}\bar{a}_{x+1} \quad \begin{matrix} x = 0, 1, \ldots \\ n = 0, 1, 2, \ldots \end{matrix}$

${}_{n|}\bar{a}_\omega = 0$

5.9. $\bar{a}_{\overline{x:\overline{n}|}} = v^n\,{}_np_x\,\bar{a}_{x+n:\overline{1}|} + \bar{a}_{\overline{n}|}(1 - vp_x) + vp_x\,\bar{a}_{\overline{x+1:\overline{n}|}}$

$\bar{a}_{\overline{\omega:\overline{n}|}} = \bar{a}_{\overline{n}|}$

5.14. $\dfrac{2}{i}\,(a_{x:\overline{n}|} - {}^2a_{x:\overline{n}|}) - {}^2a_{x:\overline{n}|} - (a_{x:\overline{n}|})^2$

5.15. $\dfrac{s^{(m)}_{\overline{1}|} - 1}{d^{(m)}}\,A_x$

5.16. a. $\frac{1}{m}\sum_{h=0}^{m-1} v^{h/m}\,{}_{h/m}p_x + \frac{1}{m}\sum_{h=m}^{(y-x)m-1} v^{h/m}\,{}_{h/m}p_x$

b. $\alpha(m) - \beta(m)(1 - vp_x)$

c. $c(x) = \alpha(m) - \beta(m)(1 - vp_x), \quad d(x) = vp_x, \quad \ddot{a}^{(m)}_{y:\overline{0}|} = 0$

5.17. $\ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x), \quad {}_{n|}\ddot{a}_x - \frac{m-1}{2m}\,{}_nE_x$

5.22. a. $\alpha(m)\,\ddot{s}_{25:\overline{40}|} - \beta(m)\left(\frac{1}{{}_{40}E_{25}} - 1\right)$ b. (i) 15.038 (ii) 196.380

5.23. a. $Y = \begin{cases} (I\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (I\ddot{a})^{(m)}_{\overline{n}|} & K = n, n+1, \ldots \end{cases}$

5.24. a. $Y = \begin{cases} (D\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ (D\ddot{a})^{(m)}_{\overline{n}|} & K = n, n+1, \ldots \end{cases}$

5.25. a. $Y = \begin{cases} (I\ddot{a})^{(m)}_{\overline{K+J+1/m}|} & K = 0, 1, \ldots, n-1, \quad J = 0, 1, \ldots, m-1 \\ n\,\ddot{a}^{(m)}_{\overline{K+J+1/m}|} & K = n, n+1, \ldots, \quad J = 0, 1, \ldots, m-1 \end{cases}$

5.31. a. $\ddot{a}_x + 0.03(Ia)_x$

b. $\sum_{k=0}^{\infty} (1.03)^k\, v^k\, {}_kp_x = \ddot{a}'_x$ evaluated at interest rate $i' = \frac{i - 0.03}{1.03}$

5.32. $\int_0^n (n-t)\, v^t\, {}_tp_x\, dt$

5.33. $1{,}200\left(\frac{a^{(12)}_{30} + {}_{10|}a^{(12)}_{30} + 3\,{}_{20|}a^{(12)}_{30} + 5\,{}_{30|}a^{(12)}_{30} - 10\,{}_{40|}a^{(12)}_{30}}{{}_{40}E_{30}}\right)$

5.34. $\bar{a}_{35:\overline{25}|} - {}_{25}p_{35}\,\bar{a}_{\overline{25}|}$

5.35. $\ddot{a}_{x:\overline{n}|} - {}_np_x\,\ddot{a}_{\overline{n}|}$

5.36. $\frac{1}{12}\,\ddot{a}_{x:\overline{25}|} - \frac{25}{12}\,{}_{25}E_x$

5.38. $v^{2n}\,{}_np_x\,(1 - {}_np_x)\,\ddot{a}^2_{x+n} + v^{2n}\,{}_np_x\,\frac{{}^2A_{x+n} - A^2_{x+n}}{d^2}$

5.41. a. $\alpha(m) = 1 + \frac{m^2-1}{12m^2}\,\delta^2 + \frac{2m^4 - 5m^2 + 3}{720m^4}\,\delta^4 + \cdots$

$$\beta(m) = \frac{m-1}{2m}\left[1 + \frac{m+1}{3m}\,\delta + \frac{m(m+1)}{12m^2}\,\delta^2 + \frac{(m+1)(6m^2-4)}{360m^3}\,\delta^3 + \cdots\right]$$

b. $\alpha(\infty) = 1 + \frac{1}{12}\,\delta^2 + \frac{1}{360}\,\delta^4 + \cdots$

$$\beta(\infty) = \frac{1}{2}\left[1 + \frac{1}{3}\,\delta + \frac{1}{12}\,\delta^2 + \frac{1}{60}\,\delta^3 + \cdots\right]$$

5.44. $\frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) v^T, \frac{I}{\delta} + \left(J - \frac{I}{\delta}\right) \bar{A}_x, \left(J - \frac{I}{\delta}\right)^2 ({}^2\bar{A}_x - \bar{A}_x^2)$

5.45. a. 14.353 b. 13.350 c. 1.002

5.51. a. 488.23 b. 700.48 c. 531.77

5.53. $\ddot{a}_{55:\overline{10}|} = 7.45735$

5.54. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.56. $\bar{a}_{55:\overline{10}|} = 7.19783$

5.57. $\bar{a}_{60:\overline{10}|} = 6.46348$, $\text{Var}(X) = 1.82621$

5.58. $\ddot{a}_{65}^{(12)} = 10.13343$, $\text{Var}(Y) = 16.87662$

5.59. 10.40189

Chapter 6

6.1. 0, 0.1779

6.3. 0.303598

6.4. a. 0.02 b. 0.00857 c. 0.02885

6.6. $\frac{\mu}{\mu + 2\delta} = {}^2\bar{A}_x$

6.10.

	Annual Premiums for (35)		
Insurance	Fully Continuous	Semicontinuous	Fully Discrete
10-Year endowment	0.075128	0.072885	0.072810
30-Year endowment	0.015371	0.014894	0.014751
60-Year endowment	0.008913	0.008621	0.008374
Whole life	0.008903	0.008611	0.008362
30-Year term	0.005117	0.004958	0.004815
10-Year term	0.002669	0.002589	0.002514

6.12. $A_x = \frac{1 - r}{1 + i - r'}, P_x = \frac{1 - r}{1 + i'}$

$\ddot{a}_x = \frac{1 + i}{1 + i - r'}$

$\frac{{}^2A_x - A_x^2}{(d\ddot{a}_x)^2} = \frac{(1 - r)r}{1 + 2i + i^2 - r}$

6.13. 0.019139

6.15. 0.032868

6.16. 0.0413

6.17. With the common $(\bar{A}_{40:\overline{25}|})$ omitted from the premium symbols, $P \le P^{(2)} \le P^{(4)} \le P^{(12)} \le \bar{P}$

6.18. $\dfrac{100}{99}$

6.19. 740.93

6.21. $P(A'^{1}_{45:\overline{20}|})$ where $A'^{1}_{45:\overline{20}|}$ is the actuarial present value of a 20-year term insurance on (45) under which $b_{k+1} = \ddot{s}_{\overline{k+1}|}$

6.22. a. 11.5451, 20.4106 b. 6.3099, 25.6458

6.24. ${}_{25}P_{60}$

6.25. b. $P^{(12)}(A^{(12)}_{1\ 65:\overline{10}|}) + d^{(12)}$

6.26. $\dfrac{100{,}000}{(1.1\ \ddot{s}_{\overline{30}|} - 0.1\ \ddot{s}_{35:\overline{30}|})}$

6.27. 0.008

6.28. $\dfrac{11{,}000\ A_x + 25\ \ddot{a}_{x:\overline{20}|}}{\ddot{a}_{x:\overline{20}|} - 1.1(I_{\overline{20}|}A)_x}$

6.29. $\dfrac{2\ A_{25} - A^{1}_{25:\overline{10}|}}{2\ \ddot{a}_{25:\overline{40}|} - \ddot{a}_{25:\overline{10}|}}$

6.30. $L_1 = v^T - \bar{P}(\bar{A}_x)\bar{a}_{\overline{T}|} \equiv 1 - \left(\dfrac{1}{\bar{a}_x}\right)\bar{a}_{\overline{T}|} = L_2$

6.31. a. −0.08 b. 0.1296 c. 0.1587

6.32. $\dfrac{\bar{A}_x}{2\ddot{a}_x - \ddot{a}_{x:\overline{5}|}}$

6.33. ${}_{20}P^{(m)}(\bar{A}_x) - {}_{20}P^{(m)}(\bar{A}_x) = {}_{20}\bar{P}(\bar{A}_x)\left(\dfrac{\bar{A}^{1}_{x:\overline{20}|} - A^{(m)\ 1}_{x:\overline{20}|}}{\delta\ddot{a}^{(m)}_{x:\overline{20}|}}\right)$

6.35. a. $\dfrac{\mu}{\delta u + \bar{P}}\left(\dfrac{\delta u + \bar{P}}{\delta + \bar{P}}\right)^{\mu/\delta} - \dfrac{\bar{P}}{\delta} < u < 1$

0 elsewhere

6.36. a. $\dfrac{1}{\sqrt{3}}$

b. 0.02

Chapter 7

7.1. ${}_1V = 0.15111$ ${}_2V = 0.30809$ ${}_3V = 0.47118$ ${}_4V = 0.64067$

7.2. ${}_1V = 0.14925$ ${}_2V = 0.30492$ ${}_3V = 0.46741$ ${}_4V = 0.63712$

7.3. ${}_1V = 1.2871$ ${}_2V = 2.6996$ ${}_3V = 4.2553$ ${}_4V = 5.9748$

7.4. ${}_1V = 0.15064$ ${}_2V = 0.30730$ ${}_3V = 0.47025$ ${}_4V = 0.63980$

7.5. a. $1 = \frac{1}{5}\int_0^5 e^{0.1[1.06^{\cdot t}(1+P/\delta)-P/\delta]}\,dt$ where $\delta = \log(1.06)$

$${}_1V = 10\log\left(\frac{1}{4}\int_0^4 e^{0.1[1.06^{-t}(1+P/\delta)-P/\delta]}\,dt\right)$$ where P and δ are ∞ in (a)

b. $P = 0.388380$, ${}_1V = 0.182825$

7.6. ${}_tL = \begin{cases} v^U - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{U}|} & U < n - t \\ v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} & U \geq n - t \end{cases}$

7.7. $\mathrm{E}[{}_tL] = \bar{a}_{x+t:\overline{n-t}|}$, $\mathrm{Var}({}_tL) = \dfrac{{}^2\bar{A}_{x+t:\overline{n-t}|} - \bar{A}^2_{x+t:\overline{n-t}|}}{\delta^2}$

7.8. a. $\bar{A}_{45:\overline{20}|} - {}_{20}\bar{P}(\bar{A}_{35:\overline{30}|})\,\bar{a}_{45:\overline{10}|}$ b. $\bar{A}^1_{50:\overline{5}|}$

7.9. a. $u_0 = \dfrac{-\log(\bar{A}_x)}{\delta}$ b. 23.2476

7.10. 41.7524

7.11. $F_{{}_tL}(y) = 0$ $\qquad y < v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \frac{1}{\delta}\log\frac{\delta y + \bar{P}(\bar{A}_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} \leq y < 1$$

$F_{{}_tL}(y) = 1$ $\qquad y \geq 1$

7.12. $F_{{}_tL}(y) = 0$ $\qquad y < \bar{P}(\bar{A}^1_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}(n)}{1 - F_{T(x)}(t)} = \frac{{}_np_x}{{}_tp_x} \qquad -\bar{P}(\bar{A}_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|} \leq y < v^{n-t} - \bar{P}(\bar{A}^1_{x:\overline{n}|})\,\bar{a}_{\overline{n-t}|}$$

$$F_{{}_tL}(y) = \frac{1 - F_{T(x)}\left(t - \frac{1}{\delta}\log\frac{\delta y + \bar{P}(\bar{A}^1_{x:\overline{n}|})}{\delta + \bar{P}(\bar{A}^1_{x:\overline{n}|})}\right)}{1 - F_{T(x)}(t)} \qquad v^{n-t} - \bar{P}(\bar{A}^1_{x:\overline{n}|}) \leq y < 1$$

7.14. $\bar{A}_{50} - {}_{20}\bar{P}(\bar{A}_{40})\,\bar{a}_{50:\overline{10}|}$, $[{}_{10}P(A_{50}) - {}_{20}P(A_{40})]\,\bar{a}_{50:\overline{10}|}$,

$$\left[1 - \frac{{}_{20}\bar{P}(\bar{A}_{40})}{{}_{10}\bar{P}(\bar{A}_{50})}\right]\bar{A}_{50},\ {}_{20}\bar{P}(\bar{A}_{40})\,\bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40}$$

7.15. $\bar{A}_{50:\overline{10}|} - \bar{P}(\bar{A}_{40:\overline{20}|})\,\bar{a}_{50:\overline{10}|}$, $[\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})]\,\bar{a}_{50:\overline{10}|}$,

$$\left[1 - \frac{\bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|})}\right]\bar{A}_{50:\overline{10}|},\ \bar{P}(\bar{A}_{40:\overline{20}|})\,\bar{s}_{40:\overline{10}|} - {}_{10}\bar{k}_{40},$$

$$1 - \frac{\bar{a}_{50:\overline{10}|}}{\bar{a}_{40:\overline{20}|}},\ \frac{\bar{P}(\bar{A}_{50:\overline{10}|}) - \bar{P}(\bar{A}_{40:\overline{20}|})}{\bar{P}(\bar{A}_{50:\overline{10}|}) + \delta},\ \frac{\bar{A}_{50:\overline{10}|} - \bar{A}_{40:\overline{20}|}}{1 - \bar{A}_{40:\overline{20}|}}$$

7.16. $\bar{P}(_{30|}\bar{a}_{35})\,\bar{s}_{35:\overline{20}|}$

7.18. (7.3.3)

7.19. $A_{50} - {}_{20}P_{40}\,\ddot{a}_{50:\overline{10}|}$, $({}_{10}P_{50} - {}_{20}P_{40})\,\ddot{a}_{50:\overline{10}|}$, $\left(1 - \frac{{}_{20}P_{40}}{{}_{10}P_{50}}\right)A_{50}$,

${}_{20}P_{40}\,\ddot{s}_{40:\overline{10}|} - {}_{10}k_{40}$

7.20. $A_{50:\overline{10}|}$ − $P_{40:\overline{20}|}$ $\ddot{a}_{50:\overline{10}|}$, $(P_{50:\overline{10}|} - P_{40:\overline{20}|})\ \ddot{a}_{50:\overline{10}|}$,

$$\left(1 - \frac{P_{40:\overline{20}|}}{P_{50:\overline{10}|}}\right) A_{50:\overline{10}|},\ P_{40:\overline{20}|}\ \ddot{s}_{40:\overline{10}|} - {}_{10}k_{40},$$

$$1 - \frac{\ddot{a}_{50:\overline{10}|}}{\ddot{a}_{40:\overline{20}|}},\ \frac{P_{50:\overline{10}|} - P_{40:\overline{20}|}}{P_{50:\overline{10}|} + d},\ \frac{A_{50:\overline{10}|} - A_{40:\overline{20}|}}{1 - A_{40:\overline{20}|}}$$

7.22. $\frac{1}{5}$

7.23.

Insurance	Fully Continuous	Semicontinuous	Fully Discrete
30-Year endowment	0.17530	0.17504	0.17407
Whole life	0.08604	0.08566	0.08319
30-Year term	0.03379	0.03370	0.03273

7.24. (b) and (c)

7.26. All but (d)

7.27. All

7.29. 0.008

7.30. 0.240

7.31. a. 0.005527
b. 0.051255
c. 0.946122
d. 0.132109

7.32. a. 0.0241821 b. 0.0189660

Chapter 8

8.1. a. $\frac{1 - r}{1 + i}$

b. $\frac{(1 - r)(1 + i + r)}{(1 + i)(1 + i - r)}$

8.2. $\frac{\int_0^\infty b_t\, v^t\, {}_tp_x\, \mu_x(t)\, dt}{\int_0^\infty w(t)\, v^t\, {}_tp_x\, dt}$

8.3. a. $\frac{\mu}{\delta + \mu}$

b. $\frac{\mu t}{\delta + \mu}$

8.5. a. $(P_{x+1} - vq_{x+h})\ {}_hp_x\ vq_x$

b. $(P_{x+1} - vq_{x+h})\ {}_hp_x(v\ {}_jp_x\ q_{x+j} + {}_jq_x\ P_{x+1})$

c. If $P_{x+1} - vq_{x+n} < 0$, then $\text{Cov}(C_j, C_h) < 0$ for all $j < h$

8.6. If $1 - vq_{x+h}\ \ddot{s}_{\overline{h+1}|} < 0$, then $\text{Cov}(C_j, C_h) < 0$ for all $j < h$

8.13. $(\bar{A}_{x:\overline{40}|})$ is omitted from the reserve and premium symbols

a. $\frac{1}{2}\ {}_{20}V + \frac{1}{2}\ {}_{21}V + \frac{1}{2}\ P$ b. $\frac{1}{2}\ {}_{20}\bar{V} + \frac{1}{2}\ {}_{21}\bar{V}$

c. $\frac{1}{2}\ {}_{20}V^{(2)} + \frac{1}{2}\ {}_{21}V^{(2)}$ d. $\frac{1}{3}\ {}_{20}V^{(2)} + \frac{2}{3}\ {}_{21}V^{(2)} + \frac{1}{3}\ P^{(2)}$

e. Same as (b) f. $\frac{1}{3}\ {}_{20}\bar{V} + \frac{2}{3}\ {}_{21}\bar{V} + \frac{1}{3}\ P^{\{2\}}$

8.14. 0.05448

8.17. b. $\text{Var}(L) = 0.076090$

8.18. a. 0.0067994 b. 0.1858077 c. 0.2012024
d. 0.0275369 e. 0.0255406

8.21. $-{}_tp_x[\delta\ {}_t\bar{V}(\bar{A}_x) + \bar{P}(\bar{A}_x)]$

8.22. a. ${}_tp_x\ [\pi_t + \delta_t\bar{V} - b_t\ \mu_x(t)]$

b. $v^t[\pi_t + \mu_x(t)\ {}_t\bar{V} - b_t\ \mu_x(t)]$

c. $v^t\ {}_tp_x\ [\pi_t - b_t\ \mu_x(t)]$

8.26. a. and b. 1,491.03 c. 343.84 d. 0

8.27. a. 1,490,915

b. 6,450,962; 1,495,093, which is 1.00280 times the reserve

c. 5,311,375; supplement is 3,791, which is 0.00254 times the reserve

d. For b.: 645,096,250; 149,133,281, which is 1.00028 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00025 times the reserve

8.28. a. 1,104,260 is the reserve for these policies

b. 6,450,962; 1,108,438, which is 1.00378 times the reserve

c. 5,311,375; supplement is 3,791, which is 0.00343 times the reserve

d. For b.: 645,096,250; 110,467,781, which is 1.00038 times the reserve
For c.: 531,137,500; supplement is 37,911, which is 0.00034 times the reserve

8.29. $5{,}000[{}_{10}\bar{V}(\bar{A}_{30}) + P^{\{1\}}(\bar{A}_{30}) + {}_{11}\bar{V}(\bar{A}_{30})]$

8.30. a. 0.2 b. 0.25 c. 0.7584 d. 0.27

8.32. 0.081467

8.34. a. 355.6563

b. 2,614.2511